To

Mr Bye

from his affectionate form, on

account of his election to

Tame Grammar School

FIRST THREE SECTIONS OF THE FIRST BOOK

OF

NEWTON'S PRINCIPIA.

NEWTON'S PRINCIPIA,

FIRST BOOK, SECTIONS I., II., III.,

WITH

NOTES AND ILLUSTRATIONS,

AND A

COLLECTION OF PROBLEMS

PRINCIPALLY INTENDED AS EXAMPLES OF NEWTON'S METHODS.

BY

PERCIVAL FROST, M.A.,

FORMERLY FELLOW OF ST. JOHN'S COLLEGE;
MATHEMATICAL LECTURER OF KING'S COLLEGE.

Principiis enim cognitis, multo facilius extrema intelligetis.—CICERO.

PREFACE.

In publishing the following work my principal intention is to explain difficulties which may be encountered by the student on first reading the *Principia*, and to illustrate the advantages of a careful study of the methods employed by Newton, by shewing the extent to which they may be applied in the solution of problems. I have also endeavoured to give assistance to the student who is engaged in the study of the higher branches of Mathematics, by representing in a geometrical form several of the processes employed in the Differential and Integral Calculus, and in the analytical investigations of Dynamics.

In my version of the first section and the beginning of the second I have adhered as closely as I could to the original form; and, in the cases in which sections have been interpolated, or the form of demonstration changed, I have indicated such changes and interpolations by brackets.

It is generally advisable not to deviate from Newton's words in the demonstrations of the Lemmas; but in many cases, I suppose purposely, he expressed himself very concisely, as in Lemmas IV. and X., and he was contented with simply giving the enunciation of Lemma V.; therefore in these cases interpolations have been made which, I believe, are in accordance with Newton's plan of demonstration.

Throughout the Problems and Theorems which depend upon the sixth proposition, the variations are replaced by equations. By this method of treating the subject I conceive that clearer ideas of the meaning of each step are obtained by the student.

In this edition I have introduced some notes on the geometrical solution of some problems relating to maxima and minima, and I have placed the investigations of the properties of the curves, which, after the conic sections, are the best examples for illustrating geometrical methods, in a more prominent position, at the end of the first section.

I have derived great assistance in the preparation of my notes from the study of Whewell's *Method of Limits*, and from several early editions of *Newton*, especially that of Carr.

With respect to the three Laws of Motion, I may remark that I have not commenced the work by enunciating and making observations upon them, partly because I should only have been repeating

what has been said so well by Thompson, Tait, and Maxwell, whose works are in everybody's hands, and partly because in the course of reading recommended to students, for whose benefit my work was especially intended, those laws will have been already discussed in the elementary treatises on Dynamics.

The Problems are principally selected from the papers set in the Mathematical Tripos, and in the course of the College Examinations, and I have generally divided them into two portions, the first of which contains those problems which are capable of solution by more direct applications of the propositions which they illustrate, and are within the powers of a larger number of students In both portions I have been careful to introduce very few problems which are not capable of solution by methods given in the work.

At the end of the work I have given hints for the solution, and in many cases complete solutions, of the problems; and in doing so I am acting in direct opposition to my previously expressed opinion, but additional experience of fifteen years has shewn me that it a satisfaction to a student who has not been able to solve a problem to see a solution of it; and, even when he has been successful, to compare his solution with that of an older hand. The principal objection to the publication of solutions

is that they are frequently referred to prematurely; but a wise student will treat them only as a *dernier ressort*.

In solving the Problems I have noticed two errors which should be corrected as follows:

XIII. 12. ....half the chord....is the harmonic mean, &c.

XXVIII. 6. ....velocity in a circle whose radius is the length of the unstretched string, &c.

Two sets of Problems have been numbered XXVII., the second is written XXVII. *bis* in the Solutions.

I take this opportunity to express my thanks to Mr. Stearn, of King's College, for his kindness in correcting the errors of the press and for many valuable suggestions.

<div align="right">PERCIVAL FROST.</div>

CAMBRIDGE,
    February, 1878.

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NEWTON'S FIRST BOOK

CONCERNING THE MOTION OF BODIES.

SECTION I.

ON THE METHOD OF PRIME AND ULTIMATE RATIOS·

LEMMA I.

Quantities, and the ratio of quantities, which, in any finite time, tend constantly to equality, and which, before the end of that time, approach nearer to each other than by any assigned difference, become ultimately equal.

If not, let them become ultimately unequal, and let their ultimate difference be *D*. Hence [since, throughout the time, they tend constantly to equality], they cannot approach nearer to each other than by the difference *D*, contrary to the hypothesis [that they approach nearer than by any assigned difference. Therefore, they do not become ultimately unequal, that is, they become ultimately equal].

Variable Quantities.

1. The *Quantities*, of which Newton treats in this Lemma, are variable magnitudes, described by a supposed law of construction, the variation of these magnitudes being due to the arbitrary progressive change of some element of the construction employed in the statement of the law.

When, in the progressive change of this element, it receives the last value which is assigned to it in any proposition, the hypothesis is said to arrive at its ultimate form, or to be indefinitely extended.

Thus, let ABP be a semicircle, ACB its diameter, BP any arc, PM the ordinate perpendicular to ACB, then, as the arc BP gradually diminishes, AM is a variable magnitude, continually increasing, and BP is the element of the construction,
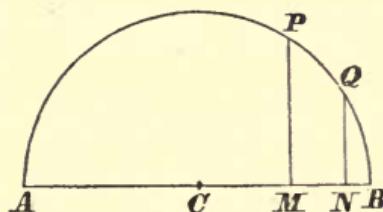
to the arbitrary change of which the variation of AM is due; and if BP may be made as small as we please, AM may be made to approach to AB nearer than by any difference that can be named, and the hypothesis approaches its ultimate form.

Again, if ABC be a triangle, and AB be divided into a number of equal portions, Aa, ab, bc, ..., and a series of parallelograms be inscribed upon those bases, whose sides $a\alpha$, $b\beta$, $c\gamma$, ... are parallel to BC and terminated in AC, the sum of the areas of the parallelograms will be a variable magnitude, defined by that construction, and changing in a progressive manner, if the
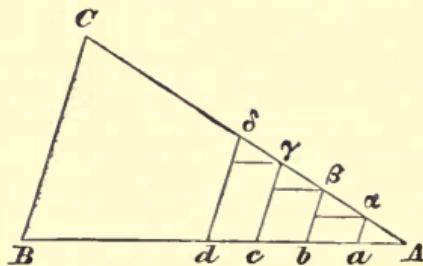
number of parts into which AB is divided be continually increased. In this case the number of parts is the variable element of the construction. In the ultimate form of the hypothesis, it will be shewn, Lemma II., that the sum of the parallelograms is the area of the triangle when the number is increased indefinitely.

2. The variation of a magnitude is *continuous*, when in the passage from *any* one value to *any* other, throughout its change,

it receives every intermediate value, without becoming infinite. When this is not the case, the variation is *discontinuous*.

According to the hypothesis in the last illustration, the number of parts into which AB is divided being exact, the magnitude varies discontinuously, *i.e.* the sum of the areas does not pass through all the intermediate values between any two states of the progress.

If the hypothesis be changed, equal portions being set off commencing from B, and Aa remaining over and above after ba, the last of the portions for which there is room, these equal portions could be made to diminish gradually, and the sum of the areas would in that case vary continuously.

Tendency to Equality.

3. Quantities are ultimately equal, when they are ultimately in a ratio of equality.

4. Quantities, which always remain *finite*, throughout the change of the hypothesis by which they are described, tend continually to equality, when their difference continually diminishes.

Thus, in fig. 1, page 2, let BQ be an arc, always in a given ratio to BP, and let QN be the corresponding ordinate; a BP continually diminishes, AM and AN remain finite, and, since their difference continually diminishes, they tend continually to equality.

5. Quantities, which may become *indefinitely small*, or *indefinitely great*, as the hypothesis is indefinitely extended, tend continually to equality, when the ratio of their difference to either of them continually diminishes.

To illustrate this test of a tendency to equality, let us suppose, in fig. 1, page 2, that the arc BP is double of the arc BQ; then, since $(\text{chd } BP)^2 = AB.BM$, and $(\text{chd } BQ)^2 = AB.BN$,

$$\therefore \quad BM : BN :: (\text{chd } BP)^2 : (\text{chd } BQ)^2$$

$$:: (\text{arc } BP)^2 : (\text{arc } BQ)^2 :: 4 : 1 \text{ ultimately,}$$

$$\therefore \quad MN : BN :: 3 : 1 \text{ ultimately;}$$

hence, we observe that BM and BN have a difference, which tends continually to become $3BN$, the ratio of which to either is finite, so that, although both tend to become indefinitely small as the hypothesis tends to its ultimate form, BM and BN do not satisfy the condition requisite for a tendency to equality.

Observations on the Lemma.

6. We will now proceed to examine the force of the other important terms employed in the statement of the first Lemma.

The expression " in any finite time " (tempore quovis finito), signifies what has been called the indefinite extension of the hypothesis from some definite state to its ultimate form.[*]

The law of the variation of the magnitudes under consideration is obtained by the examination of their construction while the element, to which the change is due, is at a finite distance from its final value, and the finite time is the supposed time occupied in the passage from this definite to the ultimate state.

In the first illustration, Art. 1, it denotes the progressive diminution of BP, from being a *finite* magnitude to the point of evanescence.

In the second, the progress from *any* finite number of equal portions to an indefinite number.

7. The expression " which *constantly* tend " (quæ constanter tendunt) signifies that, from the commencement of the *finite time* to the limit of the extension of the hypothesis, the differences continually diminish.

To illustrate this mode of expression, let BC be a quadrant
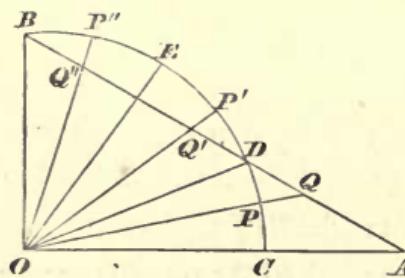
of a circle whose bounding radii are OB, OC, and let BDA be a straight line cutting the arc BDC and the radius OC in D and A, and let OP be a radius revolving from OC to OB, and cutting BA in Q, E the point of bisection of the arc BD.

OP and OQ twice tend *to* equality, viz. from OC to OD and from OE to OB, and once *from* equality from OD to OE; it is only from OE to OB that OP'' and OQ'' tend to equality constantly during the progress, and it is from some position between OE and OB that the finite time must be considered to commence.

8. "Before the end of that time" (ante finem temporis) implies that, however small the given difference may be, a less difference than that difference is arrived at, while the distance from the ultimate state is still finite, however near to the final state it may be necessary to proceed.

Thus, if, in the last figure, the angle BOD be 60°, the radius one inch, and the given difference $\frac{36}{100000}$ or $\frac{18}{100000}$ of an inch, the difference PQ will be less than the given difference, if the revolving radius be 2' or 1', respectively, from the ultimate position; and so on, however small we choose the difference.

9. In the proof of the Lemma, if the ultimate difference be D, the quantities cannot approach nearer than by that given difference; otherwise, they would, in one part of the progression, have been tending *from* equality in order to arrive ultimately at that difference, contrary to the statement of the proposition in the words "ad æqualitatem constanter tendunt."

The nature of the proof, which is more difficult than may at first sight appear, can be illustrated as follows, by examining the effect of the omission of some of the points in the statement of the Lemma.

Draw Oy, Ox at right angles, AB any straight line meeting Oy in A, CED a curve touching AB in E and meeting Oy in C, CD' another touching a straight line parallel to AB in C, $MQPP'$ a common ordinate.

As OM diminishes until it becomes indefinitely small, $MQPP'$ moves up to Oy.

In both curves, the ordinates MQ and MP or MP' have an ultimate difference CA, equal to D suppose.
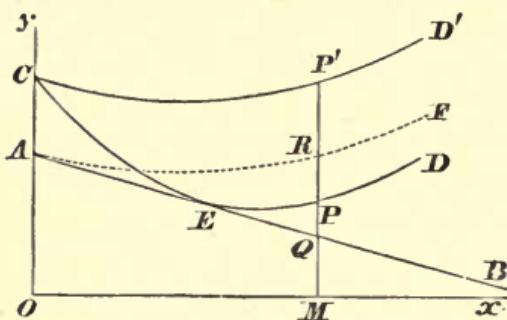
Omit the word " constanter," and the curve CED is admissible in a representation of the approach of the quantities; because the ordinates approach, before the end of the time, nearer than by any assignable difference, as at E, although the condition of continual tendency to equality is not satisfied.

Omit the words " ante finem temporis," and CD' will be sufficient; for, in this case, they tend continually to equality, but before the end of the time they do not approach nearer than by any assignable difference, and they are ultimately unequal.

In the case of the dotted line ARF touching AB at A, all the conditions are satisfied. QM and RM tend continually to equality, and their difference may be made less than any given difference before OM vanishes.

Limit of a Variable Quantity.

10. When a variable quantity tends continually to equality with a certain fixed quantity, and approaches nearer to this quantity than by any assignable difference, as the hypothesis determining its variation is approaching its ultimate form, this fixed quantity is called the *Limit of the variable quantity*.

The tests are: that there should be a tendency to equality; that this tendency should be continued from some finite condition; and that the approach should, during the progression to the ultimate form, be nearer than by any assignable difference.

Thus, as is mentioned in the Scholium at the end of the

section, the variable quantity does not become equal to, or surpass the limit, before the arrival at the ultimate form.

Limiting Ratio of Variable Quantities.

11. If two quantities continually diminish or increase, and the ratio of these quantities tends continually to equality with a certain fixed ratio, and may be made to differ from that ratio by less than any assignable difference, as the hypothesis determining their variation is indefinitely extended, this fixed ratio is called the *limiting ratio* of the varying quantities.

Ultimate Ratio of Vanishing Quantities.

12. When the ultimate form of the hypothesis brings the quantities to a state of evanescence, they are called *vanishing quantities ;* and the limiting ratio, or the limit of the ratio, is the *ultimate ratio of the vanishing quantities.*

The expression "vanishing quantities" does not imply that the quantities *are* indefinitely small while under examination, but only that they *will be* so in the ultimate form; which observation implies that *the ratio of the vanishing quantities* is not an equivalent expression with the *ultimate ratio of the vanishing quantities*, the former being taken "ante finem temporis."

"Ultimæ rationes illæ quibuscum quantitates evanescunt, revera non sunt rationes quantitatum ultimarum." See Scholium, at the end of the section.

Thus, let GC, FC be two straight lines intersecting AB in G, F, and draw ADE, MPQ, perpendicular to AB.

Let α, β be the areas $AMPD$, $AMQE$, then it is easily found
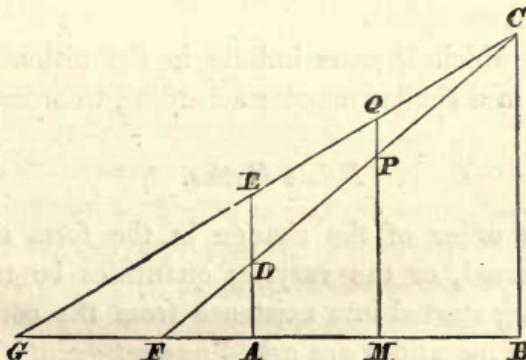
that $\alpha : \beta :: AD + MP : AE + MQ$; now, let MPQ be supposed to move up to ADE, then, in the ultimate form of the hypothesis, α and β vanish, and are called vanishing quantities from this circumstance.

Also, the *ultimate ratio* of the vanishing quantities is $AD : AE$.

In this case, since $MP : MQ$ is not equal to $AD : AE$, the ratio of the vanishing quantities, viz. $AD + MP : AE + MQ$, is different from $AD : AE$, the ultimate ratio.

Orders of Vanishing Quantities.

13. When we have to consider various kinds of vanishing quantities, it is necessary to consider their relative magnitudes, and for this purpose if one of them be selected as a standard of small quantities, this quantity, and all the vanishing quantities of which the ultimate ratio to it is finite, are called vanishing quantities *of the first order*.

If α, β be any two vanishing quantities, and $\beta : \alpha$ vanish in the limit, β is said to be a vanishing quantity of a higher order than α.

If α be of the first order, and $\beta : \alpha^2$ be ultimately finite, β is called a vanishing quantity *of the second order*, and so on for higher orders.

Trigonometrical functions give familiar illustrations of these orders; let θ be taken as the standard of vanishing quantities; $\sin \theta$ $\tan 2\theta$, $\sin \frac{1}{2}\theta$ are all of the first order, since their ratios to θ are ultimately 1, 2 and $\frac{1}{2}$; $\mathrm{vers}\,\theta$, which is equal to $2 \sin^2 \frac{1}{2}\theta$ is of the second order, $\tan \theta - \theta$ and $\theta - \sin \theta$ are of the third order.

Quantities which become infinite in the ultimate state are also classified in a similar manner according to orders.

Prime Ratios.

14. If the order of the change in the form of the hypothesis be reversed, or the varying quantities be tending from equality, having started into existence from the commencement of the time, the quantities are called *nascent quantities ;* and the

ratio with which they commence existence is called the *prime ratio* of the nascent quantities.

Application of Lemma I to the investigation of certain Limits.

(1) *Limit of* $\dfrac{1+x}{2-x}$, *as* x *gradually diminishes, and ultimately vanishes.*

Since the difference between $\dfrac{1+x}{2-x}$ and $\dfrac{1}{2}$ is $\dfrac{3x}{2(2-x)}$, this difference continually diminishes as x gradually diminishes, and, by diminishing x sufficiently, may be made less than any assignable difference.

Hence, $\dfrac{1+x}{2-x}$ will tend continually to equality with $\frac{1}{2}$, if we commence from some value of x less than 2, and the difference may be made less than any assignable quantity *ante finem temporis,* therefore $\frac{1}{2}$ satisfies all the conditions of being the required limit.

(2) *Limit of* $\dfrac{2+x}{5+3x}$, *when* x *increases indefinitely.*

Since the difference $\dfrac{2+x}{5+3x} - \dfrac{1}{3} = \dfrac{1}{3(5+3x)}$, which continually diminishes as x increases, and may be made less than any assignable difference; therefore, as before, $\frac{1}{3}$ satisfies all the conditions of being a limit of $\dfrac{2+x}{5+3x}$.

(3) *Tangents are drawn to a circular arc, at its middle point, and at its extremities. Shew that, when the arc diminishes, the area of the triangle formed by the chord of the arc, and the two tangents at the extremities, is ultimately four times that of the triangle formed by the three tangents.*

Let C be the middle point of the arc, AB the chord, FA, FB, DCE the three tangents, and O the centre of the circle,

$$\triangle FDE : \triangle FAB : FC^2 : FG^2.$$

Now $FC(FC + 2CO) = FA^2 = FO.FG$;

$$\therefore \quad FC : FG :: FO : FC + 2CO;$$
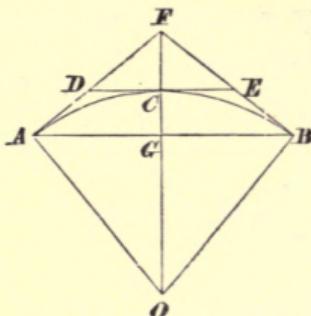
therefore, since FC vanishes in the limit, $FC : FG :: CO : 2CO$
and $FG = 2FC$, ultimately;

$$\therefore \quad \triangle FDE : \triangle FAB :: 1 : 4.$$

(4) *Limit of* $\dfrac{x^m - 1}{x - 1}$, *when* x *differs from* 1 *by an indefinitely*
small quantity, m *being any number, integral or fractional, posi-*
tive or negative.

First, where m is a positive whole number,

$$\frac{x^m - 1}{x - 1} = x^{m-1} + x^{m-2} + \ldots + x + 1,$$

which may be made to differ from m by less than any assignable
difference by taking x sufficiently near to unity.

Next, let $m = \dfrac{p - q}{r}$, p, q, and r being positive whole
numbers, and let $x = y^r$;

$$\therefore \; \frac{x^m - 1}{x - 1} = \frac{y^{p-q} - 1}{y^r - 1} = \frac{1}{y^q} \cdot \frac{y^p - y^q}{y^r - 1} = \frac{1}{y^q} \cdot \frac{y^p - 1 - (y^q - 1)}{y^r - 1}$$

$$= \frac{1}{y^q} \cdot \frac{\dfrac{y^p - 1}{y - 1} - \dfrac{y^q - 1}{y - 1}}{\dfrac{y^r - 1}{y - 1}}.$$

This may be made to differ from $\dfrac{p - q}{r}$ or m by a quantity
less than any assignable quantity by taking x, and therefore y,
sufficiently near to unity; hence, whether it be integral or
fractional, positive or negative, m is the limit required.

When we divide the numerator and denominator by $y - 1$,
y is not equal to 1, the time chosen being *ante finem temporis*

while the difference is finite. See the direction in the Scholium referred to above: " Cave intelligas quantitates magnitudine determinatas, sed cogita semper diminuendas sine limite."

(5) *Limit of* $\dfrac{1^p + 2^p + 3^p + \ldots + n^p}{n^{p+1}}$, *when n is indefinitely increased, p being any positive number.*

Since this sum is the arithmetic mean of the n fractions

$$\left(\frac{1}{n}\right)^p, \quad \left(\frac{2}{n}\right)^p, \quad \ldots \left(\frac{n}{n}\right)^p,$$

therefore, for all positive values of p, integral or fractional, it lies between $\left(\dfrac{1}{n}\right)^p$ and $\left(\dfrac{n}{n}\right)^p$ or 1, therefore its ultimate value lies between 0 and 1.

This being an important limit, we will investigate it first for the particular case in which p is integral and positive, and then generally when p is any positive quantity.

$$\text{Let } S_n = 1^p + 2^p + \ldots + n^p;$$
$$\text{then } S_{n+1} = 1^p + 2^p + \ldots + n^p + (n+1)^p;$$
$$\therefore \ S_{n+1} - S_n = (n+1)^p.$$

If therefore we assume that

$$S_n = An^{p+1} + Bn^p + \ldots + Ln + M,$$
$$\text{then } S_{n+1} = A(n+1)^{p+1} + B(n+1)^p + \ldots + L(n+1) + M;$$
$$\therefore \ (n+1)^p = A\{(n+1)^{p+1} - n^{p+1}\} + B\{(n+1)^p - n^p\} + \ldots$$
$$= A\{(p+1)n^p + \tfrac{1}{2}(p+1)pn^{p-1} + \ldots\}$$
$$+ B\{pn^{p-1} + \tfrac{1}{2}p(p-1)n^{p-2} + \ldots\} + \ldots,$$

we obtain, by equating the coefficients, $p+1$ equations for determining the values of the $p+1$ constants $A, B, \ldots L$, which reduce the equation to an identity.

The first of these equations is $1 = (p+1)A$;

$$\therefore \ S_n = \frac{1}{p+1} \cdot n^{p+1} + Bn^p + \ldots,$$
$$\text{and } \frac{S_n}{n^{p+1}} = \frac{1}{p+1} + \frac{B}{n} + \frac{C}{n^2} + \ldots + \frac{M}{n^{p+1}};$$

hence, if n be increased, since the number of the terms following $\dfrac{1}{p+1}$ is finite, we may make the difference between $\dfrac{S_n}{n^{p+1}}$ and $\dfrac{1}{p+1}$ diminish until it becomes less than any assignable quantity;

therefore $\dfrac{1}{p+1}$ is the limit required.

Next, let p be any positive quantity, and let l be the limit of

$$\frac{1^p + 2^p + \ldots + n^p}{n^{p+1}};$$

$$\therefore \ 1^p + 2^p + \ldots + n^p = ln^{p+1} + Bn^\beta + Cn^\gamma + \ldots,$$

in which $p+1, \beta, \gamma \ldots$ are in descending order, and $\dfrac{Bn^\beta + Cn^\gamma + \ldots}{n^{p+1}}$ vanishes, when n is made infinitely large;

$$\therefore \ 1^p + 2^p + \ldots + (n+1)^p = l(n+1)^{p+1} + B(n+1)^\beta + \ldots;$$

$$\therefore \ (n+1)^p = l\{(n+1)^{p+1} - n^{p+1}\} + B\{(n+1)^\beta - n^\beta\} + \ldots;$$

$$\therefore \ \left(1 + \frac{1}{n}\right)^p = l \cdot \frac{\left(1 + \dfrac{1}{n}\right)^{p+1} - 1}{1 + \dfrac{1}{n} - 1} + \frac{Bn^\beta}{n^{p+1}} \cdot \frac{\left(1 + \dfrac{1}{n}\right)^\beta - 1}{1 + \dfrac{1}{n} - 1} + \ldots;$$

therefore, observing that, when n is increased indefinitely,

$$\frac{\left(1 + \dfrac{1}{n}\right)^q - 1}{1 + \dfrac{1}{n} - 1} = q,$$

$$1 = (p+1)\,l + \text{limit of } \frac{\beta(1+\varepsilon)Bn^\beta + \gamma(1+\varepsilon')Cn^\gamma + \ldots}{n^{p+1}};$$

where $\varepsilon, \varepsilon', \ldots$ vanish ultimately. Let $\varepsilon_{\prime}$ be the greatest of the quantities $\varepsilon, \varepsilon', \ldots$, and let all the terms be positive, then

$$\beta(1+\varepsilon)Bn^\beta + \ldots \text{ is less than } (1+\varepsilon_{\prime}\beta\left(Bn^\beta + \frac{\gamma}{\beta}Cn^\gamma + \ldots\right),$$

and, since $\dfrac{\gamma}{\beta}, \dfrac{\delta}{\beta} \ldots$ are each less than 1,

$$\frac{\beta(1+\varepsilon)Bn^\beta + \ldots}{n^{p+1}} \text{ is less than } (1+\varepsilon_{\prime})\beta \times \frac{Bn^\beta + Cn^\gamma + \ldots}{n^{p+1}},$$

which vanishes in the limit, hence $1 = (p+1)l$ ultimately;

therefore $\dfrac{1}{p+1}$ is the limit required.

COR. $\dfrac{1}{p+1}$ is evidently also the limit of the sum

$\dfrac{1^p + 2^p + \ldots + (n-1)^p}{n^{p+1}}$, since $\dfrac{n^p}{n^{p+1}}$ vanishes in the limit.

(6) *If a straight line of constant length slide with its extremities in two straight lines, which intersect at a given angle A, and BC, bc be two positions of the line intersecting in P, which become ultimately coincident, find the limits of the ratios $Cc : Bb$ and $PC : PB$.*

By hypothesis, $BC^2 = bc^2$,

but $BC^2 = BA^2 + CA^2 - 2BA \cdot CA \cos A$,

and $bc^2 = bA^2 + cA^2 - 2bA \cdot cA \cos A$;

$\therefore\ CA^2 - cA^2 = bA^2 - BA^2 + 2\{BA(cA + Cc) - (BA + Bb)cA\}\cos A$;

$\therefore\ Cc(CA + cA) = Bb(BA + bA) + 2(BA \cdot Cc - cA \cdot Bb)\cos A$;

$\therefore\ Cc : Bb :: BA + bA - 2cA \cos A : CA + cA - 2BA \cos A$

$ :: BA - CA \cos A : CA - BA \cos A$ ultimately.
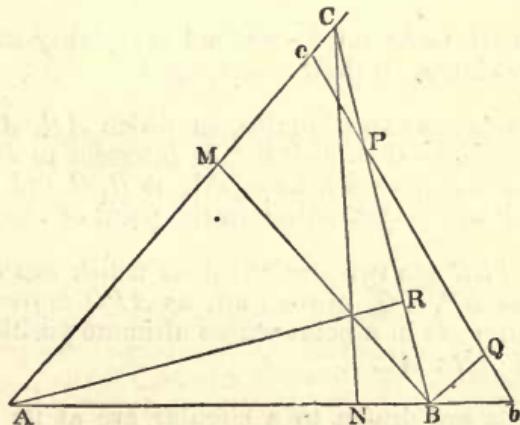
Draw CN, BM perpendicular to AB, AC, therefore the limit of the ratio $Cc : Bb$ is $BN : CM$.

Again, let BQ, drawn parallel to AC, meet bc in Q,

$$\text{then} \quad PC : PB :: Cc : BQ;$$

$$\text{also} \quad Cc : Bb :: BN : CM \text{ ultimately,}$$

$$\text{and} \quad Bb : BQ :: Ab : Ac;$$

$$\therefore \quad Cc : BQ :: BN.AB : CM.AC \text{ ultimately.}$$

Draw AR perpendicular to BC, then $BN.AB = BR.BC$ and $CM.AC = CR.BC$;

$$\therefore \quad PC : PB :: BR : CR;$$

$$\therefore \quad PC = BR \text{ and } PB = CR.$$

I.

1. Are the limits of the ratios $y^2 : x$ equal in any of the three equations

$$(1) \quad y^2 = ax^2, \quad (2) \quad y^2 = ax - b^2, \quad (3) \quad y^2 = ax - x^2,$$

when x is indefinitely diminished?

2. Find the limit of $\dfrac{x+3}{1+3x}$,

      (1) when x is indefinitely diminished,

      (2) when x is indefinitely increased.

3. Find the ultimate ratio of the vanishing quantities $ax + bx^2$, $bx + ax^2$, when x is made indefinitely small.

4. Prove that $a - bx$ and $b - ax$ tend to equality as x diminishes to zero, and yet have not their limits equal.

5. BAC, bAc are two triangles, in which AB, Ab and AC, Ac are coincident in direction, and BC, bc intersect in P; prove that, if the areas of the triangles be equal, as B, C and b, c approach, each to each, P will be ultimately in the point of bisection of BC.

6. APQ, ABC are two straight lines which are intersected by two fixed lines BP, CQ, prove that, as APQ moves up to ABC, PC and QB intersect in a point whose ultimate position divides BC in the ratio of $AB : AC$.

7. Tangents are drawn to a circular arc at its middle point, and at its extremities, and the three chords are drawn. Prove that the triangle contained by the three tangents is ultimately one-half of that contained by the three chords, when the arc is indefinitely diminished.

8. AP is a chord of a given circle, AQ a chord near AP, find the position of the point of ultimate intersection of circles described on AP, AQ as diameters, when AQ approaches to and ultimately coincides with AP.

9. A circle passes through a fixed point, and cuts off from a fixed line a chord PQ of constant length, prove that the chord of ultimate intersection of two consecutive circles bisects PQ.

10. PN is an ordinate, and PT a tangent to an ellipse, cutting the axis-major in N and T respectively; A being the vertex, shew that as P approaches A, NT is ultimately bisected in A.

11. APQ is a parabola, PM, QN ordinates to the axis AMN, with centres M and N and radii PM, QN two circles are drawn; prove that, when N approaches indefinitely near to M, if the two circles intersect, the distance of their point of intersection from PM is ultimately equal to the semi-latus rectum. What is the condition that the circles may intersect?

II.

1. What is the test of tendency to equality? If two quantities diminish so that their difference diminishes, prove that they will tend to or from equality according as the ratio of their rates of decrease is greater or less than the ratio of the greater to the less.

2. ABC is an isosceles triangle, base BC; P, Q are points on the straight lines CA, CB such that AP is always twice BQ; prove that, if PQ and AB intersect in R, and R' be the ultimate position of R, when AP is indefinitely diminished,

$$R'B : AC :: AC : 2BC \sim AC.$$

3. PMP' is a double ordinate of an ellipse, whose centre is C; R is the point of ultimate intersection of the circles described on PP' and the next consecutive double ordinate respectively, and RT is the ordinate of R. Shew that $TM : CM :: BC^2 : AC^2$. What is the condition that these circles may intersect?

4. Two concentric and coaxial ellipses have the sum of the squares of their axes equal; if the curves approach to coincidence with each other, shew that the ratio of the distances of any one of their points of intersection from the axes will be ultimately equal to the inverse ratio of the squares of the axes.

5. If a triangle be inscribed in a given circle, prove that the algebraic sum of the small variations of its sides, each divided by the cosine of the angle opposite to it, will be equal to zero.

6. *ABC*, *APQ* are drawn to cut a circle from an external point *A*; *BU*, *CT* are tangents at *B* and *C* to the circle, meeting *APQ* in *U*, *T*; shew that the ultimate ratio of *PU* : *QT*, when *APQ* moves up to *ABC*, is $AB^2 : AC^2$.

7. *BCRA* is a diameter of a circle whose centre is *C*, and *PRQ* is a chord in it perpendicular to *BA*. *PR* is bisected in *S*, and *CS* meets the circle in *S'*. If tangents at *P* and *S'* meet *BA* in *T* and *T'*, shew that when *P* moves up to *A*, $AT = 4AT'$ ultimately.

8. If the quadrilateral *ABCD* be slightly displaced in its own plane, so as to occupy the position *abCD*, and *O* be the point of intersection of *DA*, *CB*, prove that the point of ultimate intersection of *ab* and *AB* will be the foot of the perpendicular from *O* upon *AB*.

9. *PSp*, *QSq* are focal chords of a parabola, prove that, ultimately, when *P* moves up to *Q*,

$$PQ : pq :: SP^{\frac{3}{2}} : Sp^{\frac{3}{2}}.$$

10. The extremities of a straight line slide upon two given straight lines, so that the area of the triangle formed by the three straight lines is constant; find the limiting position of the chord of intersection of two consecutive positions of the circle described about that triangle.

LEMMA II.

*If, in any figure AacE, bounded by the straight lines Aa, AE
and the curve acE, any number of parallelograms Ab, Bc,
Cd, &c. be inscribed upon equal bases AB, BC, CD, &c.,
and having sides Bb, Cc, Dd, &c. parallel to the side Aa
of the figure ; and the parallelograms aKbl, bLcm, cMdn,
&c. be completed ; then, if the breadth of these parallelo-
grams be diminished, and the number increased indefi-
nitely, the ultimate ratios which the inscribed figure
AKbLcMdD, the circumscribed figure AalbmcndoE, and
the curvilinear figure AabcdE have to one another, will
be ratios of equality.*
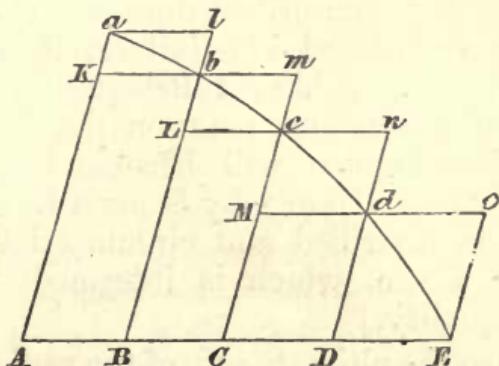
For the difference of the inscribed and circumscribed
figures is the sum of the parallelograms *Kl, Lm, Mn,
Do*, that is (since the bases of all are equal) a paral-
lelogram whose base is *Kb*, that of one of them, and
altitude the sum of their altitudes, that is, the paral-
lelogram *ABla*.   But this parallelogram, since its
breadth is diminished indefinitely [as the number of
parallelograms is increased indefinitely] becomes less
than any assignable parallelogram; therefore, by
Lemma I., the inscribed and circumscribed figures,
and, *a fortiori*, the curvilinear figure, which is inter-
mediate, become ultimately equal.

D

LEMMA III.

The same ultimate ratios are also ratios of equality, when the breadths of the parallelograms AB, BC, CD, ... are unequal, and all are diminished indefinitely.
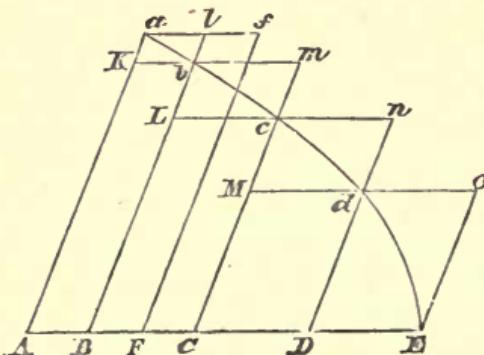
For, let *AF* be equal to the greatest breadth, and the parallelogram *FAaf* be completed. This parallelogram will be greater than the difference between the inscribed and circumscribed figures. But, when its breadth is diminished indefinitely, it will become less than any assignable parallelogram. [Therefore, *a fortiori*, the difference between the inscribed and circumscribed figures will become less than any assignable areas. Hence, by Lemma I., the ultimate ratios of the inscribed and circumscribed and the curvilinear figure, which is intermediate, will be ratios of equality.]

Cor. 1. Hence the ultimate sum of the vanishing parallelograms coincides [as to area] with the curvilinear figure.

Cor. 2. And, *a fortiori*, the rectilinear figure which is bounded by the chords of the vanishing arcs *ab*, *bc*, *cd*, &c., ultimately coincides with the curvilinear figure.

Cor. 3. As also the rectilinear circumscribed figure, which is bounded by the tangents at the extremities of the same arcs.

Cor. 4. And these ultimate figures, with respect to their perimeters *acE*, are not rectilinear figures, but curvilinear limits of rectilinear figures.

Observations on the Lemmas II. and III.

15. The statements of the propositions concerning limits of quantities and their ratios contain:

I. The hypothesis by which the quantities are defined.

II. The manner in which the hypothesis approaches its ultimate form.

III. The ultimate property when the hypothesis is thus indefinitely extended.

The strength of the proofs lies in the examination of the quantities while the hypothesis is in a finite state, before arrival at the ultimate form, and the deduction of properties by which the relations of the quantities can be pursued accurately to the ultimate state.

If in this manner we analyse the statement of Lemmas II. and III., the hypothetical constructions are given in the manner of describing the parallelograms; the extension of the hypothesis towards its ultimate form is the continual increase of the number of parallelograms *ad infinitum ;* the ultimate property is the equality of the ratio of the sums of the parallelograms and the curvilinear area.

In the proof of the Lemmas, the continual decrease of the parallelograms Al or Af shews that the conditions of ultimate equality of two quantities are all satisfied, viz., that the sums of the two series of parallelograms, since they are finite, tend continually to equality, and that they approach nearer to each other than by any assignable difference " ante finem temporis," *i.e.,* while the number of the parallelograms still remains finite.

Volumes of Revolution.

16. In a manner exactly similar to Lemma II. it may be shewn that, if Aa be perpendicular to AE, and the whole figure revolve round AE as an axis, the ultimate ratios, which the sums of the volumes of the cylinders, generated respectively by the rectangles Ab, Bc, ... and aB, bC, ... and the volume of revolution generated by the curvilinear area AEa will have to each other, will be ratios of equality.

The figure represents the cylinders generated by the inscribed rectangles.
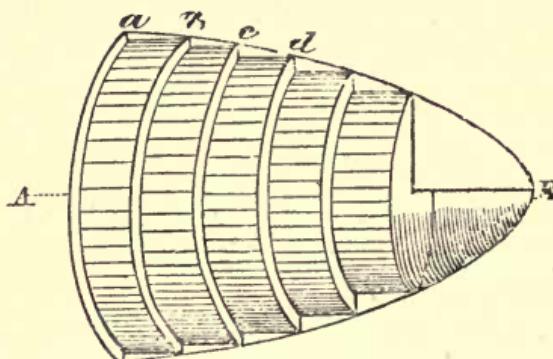
Thus the difference of the cylinders generated by Ab and aB is the annulus generated by the rectangle ab, and the difference of the two series of cylinders, which have all equal heights AB, BC, ... is the sum of such annuli, and is easily seen to be the cylinder generated by aB, which, since the height continually diminishes, may be made less than any assignable volume, hence the conditions that the two series may have the same limit are satisfied, and hence also the volume of revolution, which is greater than one sum and less than the other, is ultimately in a ratio of equality to either sum.

The same argument applies when the revolution is only through a certain angle instead of being complete, in which case the cylinders are replaced by sectors of cylindrical volumes.

Sectorial Areas.

17. The Lemmas may be extended to sectorial areas.
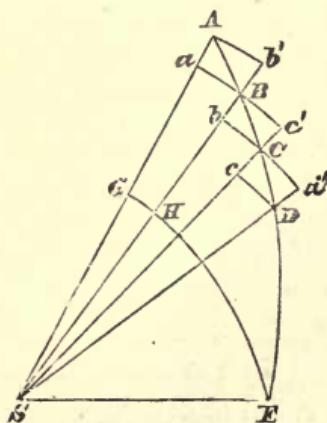
Thus, let $SABCF$ be a sectorial area, and let the angle ASF be divided into equal portions ASB, BSC, ... and the circular arcs Ab', aBc', bCd', ... be drawn with centre S; then, since the difference of the two series of circular sectors is the sum of the areas ab', bc', ..., it is equal to the difference of the greatest and least of the sectors, viz. $AGHb'$; therefore the two areas $SAb'Bc'...$ and $SaBbC...$ tend continually to equality as the number of angles is increased and their magnitudes diminished, and the ratios which these areas have to each other and to the area $SABF$ are ultimately ratios of equality.

Similarly, as in Lemma III., if ASB, BSC, ... be unequal.

Surfaces of Revolution.

18. The following proposition is the extension of the principles of the Lemmas to the determination of a method for finding the area of a surface of a solid of revolution.

Let CD be a plane curve which generates a surface of revolution by its revolution round AB, a line in its plane.

CD is divided into portions, of which PQ is one, PM, QN are perpendicular to AB; Pp, Qq are drawn parallel to AB, and each equal to PQ in length; pm, qn are perpendicular to AB. The surface generated by CD shall be the limit of the sum of the cylindrical surfaces generated by such portions as Pp or Qq.

For, the cylindrical surfaces generated by Pp and Qq are one less and the other greater than the surface generated by PQ,
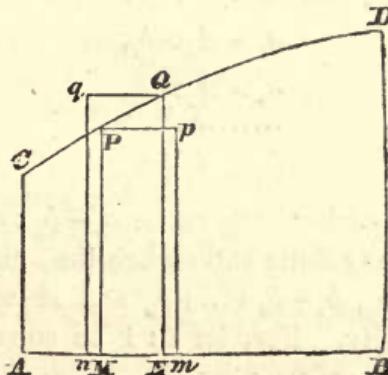
since every portion of Qq is at a greater, and every portion of Pp at a less, distance from the axis than the corresponding portions of PQ.

But these surfaces are respectively $2\pi PM . Pp$ and $2\pi QN . Qq$, and their difference is $2\pi (QN - PM) PQ$, and the ratio of this difference to the surfaces themselves is $QN - PM : PM$ or QN, which ratio is ultimately less than any given ratio.

Hence the sums of the surfaces generated by the lines corresponding to Pp and Qq have the ratio of their difference to either sum less than the greatest value of the ratio $QN - PM : PM$, which may be made less than any finite ratio. Therefore the sums of the cylindrical surfaces and the curved surface, which is intermediate in magnitude to these sums, are ultimately in a ratio of equality.

Centre of Gravity.

19. It is easily seen that the same methods are applicable to the determination of the position of the centre of gravity of any body, since it is known that, if a body be divided into any number of portions, the distance of the centre of gravity of the body from any plane is equal to the sum of the moments of all the portions divided by the sum of all the portions.

General Extension.

20. The most general extension may be stated as follows: If any magnitude A be divided into a series of magnitudes $A_1 A_2 ... A_n$, each of which, when their number is increased indefinitely, becomes indefinitely small, and two series of quantities $a_1 a_2 ... a_n$ and $b_1 b_2 ... b_n$ can be found such that

$$a_1 > A_1 > b_1,$$
$$a_2 > A_2 > b_2,$$
$$\dots\dots\dots\dots$$
$$a_n > A_n > b_n,$$

and also such that each of the ratios $a_1 - b_1 : a_1, \; a_2 - b_2 : a_2, \; ...$ becomes less than any finite ratio when the number is increased; then $a_1 + a_2 + ... + a_n, \; b_1 + b_2 + ... + b_n$ and A will be ultimately in a ratio of equality. For, let $l : 1$ be equal to the greatest of the ratios $a_1 - b_1 : a_1$, &c.;

$$\therefore \quad a_1 - b_1 + a_2 - b_2 + ... : a_1 + a_2 + ...$$

is a ratio less than $l : 1$, and may therefore be made less than

any assignable ratio by increasing the number. Therefore the two series $a_1 + a_2 + \dots$ and $b_1 + b_2 + \dots$ tend continually to equality, and the difference may be made, before the end of the time, less than any assignable magnitude; therefore the three magnitudes are ultimately in a ratio of equality.

21. COR. 1. "Omni ex parte" has not been adopted from the text of Newton, because it requires limitation, for the perimeters do not ultimately coincide with the perimeter of the curvilinear area.

In the figure for Lemma II. the perimeter of the inscribed series of parallelograms is

$$AK + Kb + bL + Lc + \dots + DA = 2AK + 2AD,$$

and the limit of this perimeter is $2Aa + 2AE$.

The perimeter of the other series of parallelograms, being $2Aa + 2AE$ is constant throughout the change, and has properly no limit.

COR. 2. The perimeter of the figure bounded by the chords ab, bc, ... ultimately coincides with that of the curvilinear figure. This coincidence will be discussed under Lemma V.

COR. 3. The same is true for the figure formed by the tangents.

COR. 4. Instead of "propterea," as in Newton, it is advisable to *state*, as in Whewell's *Doctrine of Limits*, that, if a *finite* portion of a curve be taken, and many successive points in the curve be joined so as to form a polygon, the sides of which, taken in order, are chords of portions of the curves, when the number of those points is increased indefinitely, the curve will be the limit of the polygon.

Application to the Determination of certain Areas, Volumes, &c.

(1) *Area of a parabola bounded by a diameter and an ordinate.*

Let AB, BC be the bounding abscissa and ordinate. Complete the parallelogram $ABCD$.

Let AD be divided into n equal portions, of which suppose AM to contain r, and MN to be the $(r+1)^{\text{th}}$; draw MP, NQ

parallel to AB, meeting the curve in P, Q, and Pn parallel to MN; the curvilinear area ACD is the limit of the sum of the
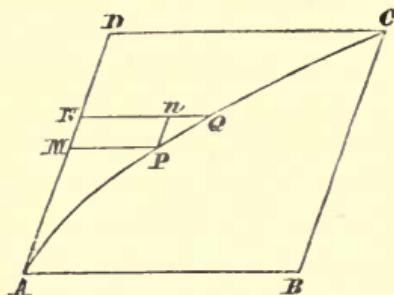
series of parallelograms constructed, as PN, on the portions corresponding to MN.

But parallelogram PN : parallelogram $ABCD$

$$:: PM.MN : CD.AD,$$

and, by the properties of the parabola,

$$PM : CD :: AM^2 : AD^2 :: r^2 : n^2,$$

$$\text{also } MN : AD :: 1 : n;$$

$$\therefore PM.MN : CD.AD :: r^2 : n^3;$$

therefore, parallelogram $PN = \dfrac{r^2}{n^3} \times$ parallelogram $ABCD$;

hence, the sum of the series of parallelograms

$$= \frac{1^2 + 2^2 + \dots + (n-1)^2}{n^3} \times \text{parallelogram } ABCD,$$

and, when the number of parallelograms is increased indefinitely,

$$\frac{1^2 + 2^2 + \dots + (n-1)^2}{n^3} = \frac{1}{3};$$

therefore, proceeding to the ultimate form of the hypothesis, the curvilinear area ACD and the parabolic area ABC will be, respectively, one-third and two-thirds of the parallelogram $ABCD$.

Note 1. If we had inscribed the series of parallelograms in ABC, AB being divided into n portions, we should have arrived at the result

$$\frac{1^{\frac{1}{2}} + 2^{\frac{1}{2}} + \dots + (n-1)^{\frac{1}{2}}}{n^{\frac{3}{2}}}$$

for the ratio of the series of parallelograms to the parallelogram *ABCD*, which might thus have been directly shewn to be ultimately $\frac{2}{3}$; but the former method is preferable, since the proof of the value of the limit depends upon simpler principles.

Note 2. If *BC* had been divided into *n* equal portions, the ratio of the parallelogram corresponding to *PN* to the parallelogram *ABCD* would have been $n^2 - r^2 : n^2$, and that of area *ABC* to parallelogram *ABCD* the limit of

$$\frac{n^2 - 1^2 + n^2 - 2^2 + \ldots + n^2 - (n-1)^2}{n^3} = 1 - \frac{1}{3} = \frac{2}{3}.$$

(2) *Volume of a paraboloid.*

Let *AKH* be the area of a parabola, cut off by the axis *AH*
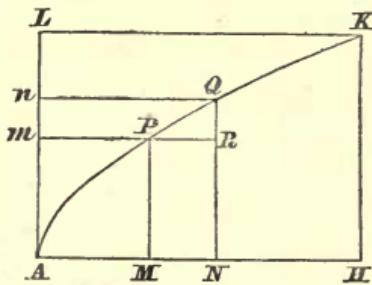
and an ordinate *HK*, which by its revolution round the axis generates a paraboloid.

Let *AH* be divided into *n* equal portions, and on *MN* the $(r+1)^{th}$, as base, let the rectangle *PRNM* be inscribed.

Cylinder generated by *PN* : cylinder by *AHKL*

$$:: PM^2 . MN : HK^2 . AH.$$
$$\text{But } PM^2 : HK^2 :: AM : AH :: r : n,$$
$$\text{and } MN : AH :: 1 : n;$$
$$\therefore PM^2 . MN : HK^2 . AH :: r : n^3.$$

Hence cylinder generated by $PN = \frac{r}{n^3} \times$ cylinder by *AHKL* ; therefore the sum of the cylinders inscribed is

$$\frac{1 + 2 + \ldots + (n-1)}{n^2} \times \text{circumscribed cylinder,}$$

and the paraboloid is the limit of the series of inscribed cylinders ; hence the volume of the paraboloid is half that of the cylinder on the same base and of the same altitude.

E

(3) *Volume of a spherical segment.*

Let AHK generate, by its revolution round the diameter AB, the spherical segment whose height is AH.
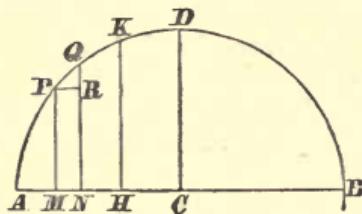
Divide AH, as before, and make the same construction;

then $PM^2 = AM.(AB-AM) = \dfrac{r}{n} AH.AB - \dfrac{r^2}{n^2} AH^2.$

Volume of cylinder generated by $PN = \pi PM^2.MN$

$$= \pi PM^2.\frac{AH}{n} = \pi AH^2.\left(\frac{r}{n^2}AB - \frac{r^2}{n^3}AH\right),$$

whence, as before, the limit of the sum

$$= \pi AH^2(\tfrac{1}{2}AB - \tfrac{1}{3}AH),$$

which is the volume proposed.

Cor. If $AH = \tfrac{1}{2}AB = AC$, the segment is a hemisphere whose volume is $\pi AC^2(AC - \tfrac{1}{3}AC) = \tfrac{2}{3}\pi AC^3$, which is two-thirds of the cylinder on the same base and of the same altitude.

(4) *Area of the surface of a right cone.*

As an illustration of the method of finding surfaces given above, suppose AHK to be a right-angled triangle, which revolves round AH, a side containing the right angle, then the hypothenuse AK generates a conical surface.

Let MN be the $(r+1)^{\text{th}}$ portion of AH, after division into
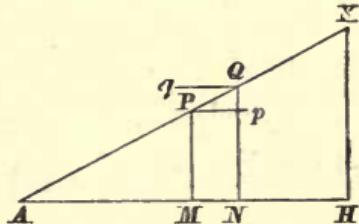
n equal portions; MP, NQ ordinates parallel to HK; Pp, Qq each equal to PQ and parallel to AH.

The areas generated by Pp and Qq respectively are

$$2\pi PM.Pp \quad \text{and} \quad 2\pi QN.Qq,$$

$$\text{and} \quad PM : HK :: AM : AH :: r : n,$$

$$QN : HK :: AN : AH :: r+1 : n,$$

$$PQ : AK :: MN : AH :: 1 : n;$$

therefore the areas are $\dfrac{r}{n^2}.2\pi HK.AK$ and $\dfrac{r+1}{n^2}.2\pi HK.AK$ respectively; and the conical surface is intermediate in magnitude between

$$2\pi HK.AK \times \frac{1+2+\ldots+(n-1)}{n^2},$$

$$\text{and} \quad 2\pi HK.AK \times \frac{1+2+\ldots+n}{n^2},$$

each of which has for its limit $\pi HK.AK$, which is therefore the area of the conical surface.

Note. The reader may notice the following method of obtaining the conical surface by development, although it is not related to the method of limits.

If a circular sector KAK', traced on paper, be cut out, the bounding radii AK, AK' can be placed in contact, so that the boundary KLK' will form a circle.

The figure so formed will be conical, AK will be the slant side, and HK in the last figure will be the radius of the circular base, whose length will be the arc of the sector KAK'.

Hence, the area of the conical surface is equal to that of the sector $KAK' = \frac{1}{2}AK.2\pi HK = \pi HK.AK$.

(5) *Mass of a rod whose density varies as the mth power of the distance from one extremity.*

Let AB be the rod, and let MN be the $(r+1)^{\text{th}}$ portion, when its length has been divided into n equal parts; and let $\rho.AM^m$ be the density at M, or the quantity of matter contained in an unit of length of the rod supposed of the same substance as the rod at the point M.

The quantity of matter in MN is intermediate between

$$\rho.AM^m.MN \quad \text{and} \quad \rho.AN^m.MN,$$

and the ratio of the difference of these to either of them is less than any assignable ratio when n is indefinitely increased.

Therefore, since $AM = \dfrac{r}{n} AB$, and $MN = \dfrac{1}{n} AB$, the mass of the whole rod is the limit of

$$\rho \cdot \frac{1^m + 2^m + \ldots + (n-1)^m}{n^{m+1}} AB^{m+1} = \frac{1}{m+1} \times \rho \cdot AB^{m+1}$$

$= \left(\dfrac{1}{m+1}\right)^{\text{th}}$ of the mass of a rod of length AB and of uniform density equal to that of the rod AB at B.

(6) *Centre of gravity of the volume of a hemisphere.*

Let CAB be a quadrant, which by its revolution round the radius CA generates the hemisphere.
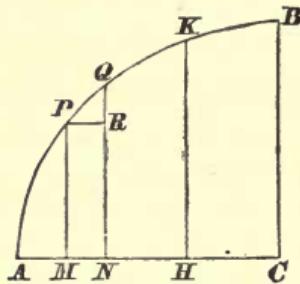
Let MR be the rectangle which generates the rth inscribed cylinder, so that $CM = \dfrac{r}{n} \times CA$ and $MN = \dfrac{1}{n} \times CA$.

If the mass of a unit of volume be chosen as the unit of mass, the mass of the cylinder generated by MR will be

$$\pi PM^2 . MN = \pi \left(CA^2 - CM^2 \right) MN = \left(1 - \frac{r^2}{n^2} \right) \pi CA^2 . \frac{CA}{n} \, ;$$

hence, the mass of the series of inscribed cylinders will be

$$\pi CA^3 - \frac{1^2 + 2^2 + \ldots + n^2}{n^3} \, \pi CA^3,$$

and the mass of the hemisphere

$$= \pi CA^3 - \tfrac{1}{3} \pi CA^3 = \tfrac{2}{3} \pi CA^3.$$

Again, the moment of the mass of the cylinder generated by MR, with respect to the base of the hemisphere, will be

$$\pi PM^2 . MN . \tfrac{1}{2} \left(CM + CN \right),$$

which differs from $\pi PM^2 . MN . CM$ by a quantity which vanishes compared with it, and is therefore ultimately $\left(\dfrac{r}{n^2} - \dfrac{r^3}{n^4} \right) \pi CA^4$; therefore the moment of the hemisphere, with respect to its base, is

$$(\tfrac{1}{2} - \tfrac{1}{4}) \pi CA^4, \text{ or } \tfrac{1}{4} \pi CA^4;$$

hence the distance of the centre of gravity of the volume of the hemisphere from C, which is the moment with respect to the base divided by the mass, is $\tfrac{3}{8} . CA$.

III.

1. Illustrate the terms "tempore quovis finito" and "constanter tendunt ad æqualitatem" employed in Lemma I. by taking the case of Lemma III. as an example.

2. Shew, from the course of the proof of Lemma II., that the ultimate ratio of vanishing quantities may be indefinitely small or great.

3. Shew that the ratio of the area of the parabolic curve, in which $PM^3 \propto AM$, to the area of the circumscribing parallelogram, of which one side is a tangent to the curve at A, is $3 : 4$.

4. Shew that the volume of a right cone is one-third of the cylinder on the same base and of the same altitude.

5. AHK is a parabolic area, AH the axis, and HK an ordinate perpendicular to the axis, $AHKL$ the circumscribing rectangle. Shew that the volumes generated by the revolution of AHK round AH, KL, AL, and HK are respectively $\tfrac{1}{2}$, $\tfrac{5}{6}$, $\tfrac{4}{5}$, and $\tfrac{8}{15}$ of the cylinder generated by the rectangle.

6. The volume of a spheroid is two-thirds of the circumscribing cylinder.

7. Find the centre of gravity of the volume of a right cone by the method of Lemma II.

8. Shew that the centre of gravity of a paraboloid of revolution is distant from the vertex two-thirds of the length of the axis.

9. Find the mass of a rod whose density varies as the distance from an extremity. Find also its centre of gravity, and shew that it is in one of the points of trisection of the rod.

10. The limiting ratio of an hyperboloid of revolution, whose axis is the transverse axis, to the circumscribing cylinder is 1 : 2 when the altitude is indefinitely diminished, and 1 : 3 when it is indefinitely increased.

IV.

1. Prove that the areas of parabolic segments, cut off by focal chords, vary as the cubes of the greatest breadths of the segments.

2. Find the mass of a circle whose density varies as the mth power of the distance from the centre.

3. Shew that the abscissa and ordinate of the centre of gravity of a parabolic area, contained between a diameter AB and ordinate BC, are $\frac{3}{5}AB$ and $\frac{3}{8}BC$ respectively.

4. A number of equal squares in one plane with their centres coincident are arranged consecutively, their sides making equal small angles, each with the adjacent ones; prove that the limit of the length of the serrated edge, when the number of squares is indefinitely increased, is equal to the circumference of a circle whose radius is a side of the square.

5. By supposing the axis of a parabola portioned off into successive lengths in the ratio 1 : 3 : 5, &c., apply Lemma III. to find the area contained by the curve and a double ordinate.

6. Find the volume generated by the revolution of an elliptic disc about an axis parallel to its major axis, and at such a given distance as not to intersect the disc.

7. In the curve ACD, BE is an ordinate perpendicular to AD, and FC is the greatest value of BE, and $\dfrac{BE}{FC} = \sin\left(\dfrac{\pi AB}{AD}\right)$.
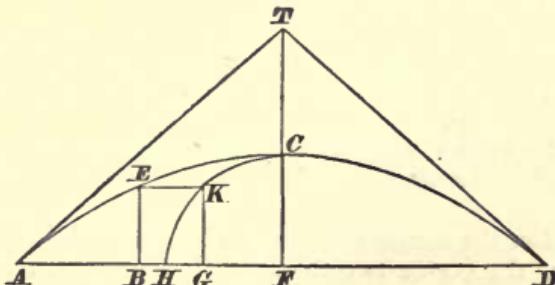
Shew that the area ABE varies as HG, where GK is the ordinate equal to BE of the circle CH, whose centre is F and radius FC.

8. In the curve of the last problem shew that the ratio of the area ACD to the triangle whose sides are AD, and the tangents AT, DT at the extremities, is $8 : \pi^2$.

9. In the curve APC, in which the relation between any rectangular ordinate PM and abscissa OM is $\dfrac{OM}{OA} = \log \dfrac{PM}{OA}$,
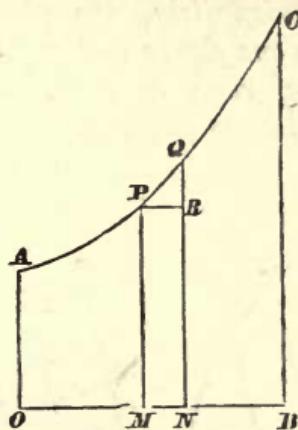
prove that the area contained between the curve, the abscissa OB, and ordinate BC, is $OA(BC - AO)$.

LEMMA IV.

If in two figures AacE, PprT there be inscribed (as in Lemmas II., III.) two series of parallelograms, the number in each series being the same, and if, when the breadths are diminished indefinitely, the ultimate ratios of the parallelograms in one figure to the parallelograms in the other be the same, each to each, then the two figures AacE, PprT will be to one another in that same ratio.
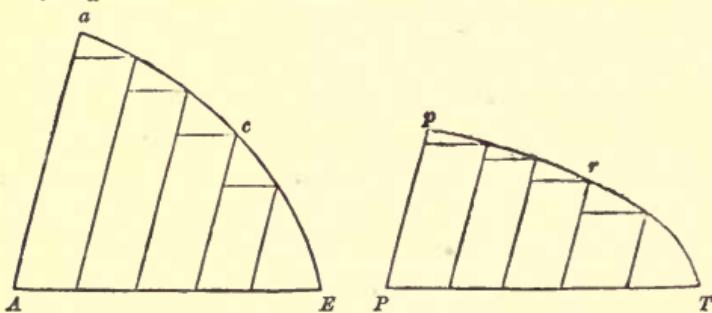
[Since the ratio, whose antecedent is the sum of the antecedents, and whose consequent is the sum of the consequents of any number of given ratios, is intermediate in magnitude between the greatest and least of the given ratios, it follows that the sum of the parallelograms described in *AacE* is to the sum in *PprT* in a ratio intermediate between the greatest and least of the ratios of the corresponding inscribed parallelograms; but the ratios of these parallelograms are ultimately the same, each to each, therefore the sums of all the parallelograms described in *AacE*, *PprT* are ultimately in the same ratio, and so the figures *AacE*, *PprT* are in that same ratio; for, by Lemma III., the former figure is to the former sum and the latter figure to the latter sum in a ratio of equality.]   Q. E. D.

Cor. Hence, if two quantities of any kind whatever be divided into any, the same, number of parts, and those parts, when their number is increased and magnitude diminished indefinitely, assume the same given ratio each to each, viz. the first to the first,

the second to the second, and so on in order, the whole quantities will be to one another in the same given ratio. For if, in the figures of this Lemma, the parallelograms be taken each to each in the same ratio as the parts, the sums of the parts will be always as the sums of the parallelograms; and, therefore, when the number of the parts and parallelograms is increased and their magnitude diminished indefinitely, the two quantities will be in the ultimate ratio of parallelogram to parallelogram, that is, (by hypothesis) in the ultimate ratio of part to part.

Observations on the Lemma.

22. The general proposition contained in the Corollary may be proved independently in the following manner:

Let A, B be two quantities of any kind, which can be divided into the same number n of parts, viz. a_1, a_2, $a_3 \ldots a_n$ and b_1, b_2, $b_3 \ldots b_n$ respectively, such that, when their number is increased and their magnitudes diminished indefinitely, they have a constant ratio $L : 1$ each to each, so that

$$a_1 : b_1 :: L(1 + \alpha_1) : 1,$$
$$a_2 : b_2 :: L(1 + \alpha_2) : 1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

where α_1, α_2, ... vanish when n is increased indefinitely.

Then, $a_1 + a_2 + \ldots : b_1 + b_2 + \ldots$ being a ratio which is intermediate between the greatest and least of these ratios, each of which is ultimately $L : 1$, we have, proceeding to the limit,

$$A : B :: L : 1;$$

that is, A and B are in the ultimate ratio of the parts.

23. The proof given in the *Principia* is as follows: "For, as the parallelograms are each to each, so, componendo, is the sum of all to the sum of all, and so the figure $AacE$ to the figure $PprT$, for, by Lemma III., the former figure is to the former sum and the latter figure to the latter sum in a ratio of equality."

The proof given in the text is substituted for this, because the demonstration breaks down for any finite distance from the ultimate form of the hypothesis.

F

Application to the determination of certain Areas, Volumes, &c.

(1) *Area of an ellipse.*

Let ACa be the major axis of an ellipse, BC the semi-minor axis, ADa the auxiliary circle, and let parallelograms be inscribed, whose sides are common ordinates to the two curves.

Let $PMNR$, $QMNU$ be any two corresponding parallelograms. The ratio of these parallelograms is $PM : QM$ or $BC : AC$.
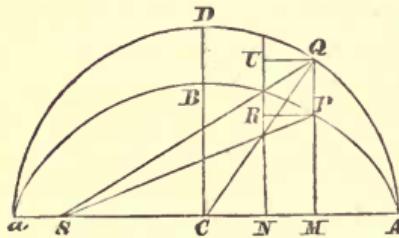
Hence, area of ellipse : area of circle :: $BC : AC$, but area of circle $= \pi AC^2$; therefore area of ellipse $= \pi AC.BC$.

(2) *Area of a sector of an ellipse, pole in the focus.*

If S be a focus of the ellipse, and SP, SQ be joined,
$$\triangle SPM : \triangle SQM :: BC : AC,$$
$$\text{and area } APM : \text{area } AQM :: BC : AC,$$
$$\text{hence, area } ASP : \text{area } ASQ :: BC : AC,$$
$$\text{but area } ASQ = \triangle SCQ + \text{sector } ACQ$$
$$= \tfrac{1}{2}SC.QM + \tfrac{1}{2}AC.\text{arc } AQ;$$
$$\therefore \text{ area } ASP = \tfrac{1}{2}\{SC.PM + BC.\text{arc } AQ\}.$$

(3) *Area of a parabolic curve cut off by a diameter and an ordinate to the diameter.*

In the following investigation it is asserted that when a chord PQ is drawn to a curve from a point P, as Q moves up to P, PQ assumes as its limiting position that of the tangent at P, which is deducible from the idea of a tangent being in the direction of the curve at the point of contact.

Let AB, BC be the diameter and ordinate; AD the tangent at A; CD parallel to AB; P, Q points near each other; PM, QN and Pm, Qn parallel respectively to AD and AB.

Let QP produced meet BA in T, and complete the parallelograms $TAmS$, $TAnU$.

Then, since QP is ultimately a tangent at P, $AT = AM$ ultimately, and the parallelogram PU is ultimately double of
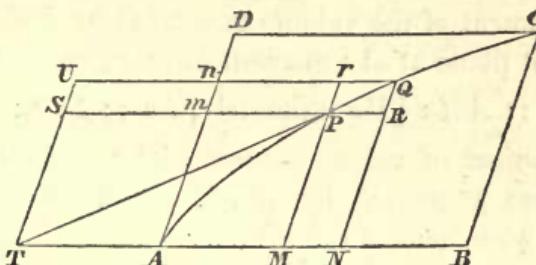
the parallelogram Pn, and the complements PN, PU are equal; therefore the parallelograms PN, Pn are ultimately in the ratio $2 : 1$.

Hence, in the curvilinear areas ABC, ACD two sets of parallelograms can be inscribed which are ultimately in the ratio $2 : 1$, each to each; therefore area ABC is ultimately double of area ACD, and is therefore two-thirds of $ABCD$.

(4) *Volume of a paraboloid of revolution.*

Let AH be the axis of the parabola APK, $AHKL$ the circumscribing rectangle. Also let PN, Pn be rectangles inscribed in the portions AHK, AKL.

Volume generated by $PN = \pi PM^2 . MN = \pi . PM . PN$.

Volume generated by $Pn = \pi QN^2 . AM - \pi PM^2 . AM$
$= \pi AM . (QN + PM) . mn = \pi (QN + PM) . Pn$;
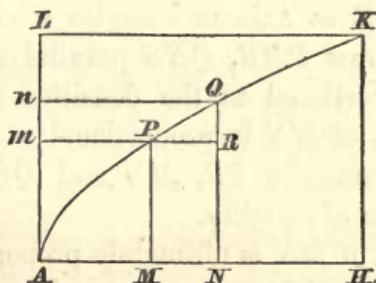
$\therefore$ vol. by PN : vol. by Pn :: $PM.PN$: $(QN + PM).Pn$, but $QN + PM = 2PM$ and $PN = 2Pn$, as in (3), and therefore vol. by $PN =$ vol. by Pn ultimately; hence, by Cor., Lemma IV., the volume of the paraboloid generated by AHK is half the volume of the circumscribing cylinder generated by AKL.

(5) *Centre of gravity of a paraboloid of revolution.*

Since the volumes generated by PN and Pn are ultimately equal, the moment of the volume generated by PN with respect to the tangent plane at A : moment of that generated by Pn

$$:: AM : \tfrac{1}{2}Pm \text{ ultimately, } i.e. :: 2 : 1 ;$$

hence the moment of volume generated by AHK is twice that of the volume generated by AKL, and the moment of the paraboloid $= \tfrac{2}{3}$ moment of the cylinder

$= \tfrac{2}{3}$ volume of cylinder $\times \tfrac{1}{2}AH = \tfrac{2}{3}$ volume of paraboloid $\times AH$;

hence the distance of the centre of gravity of the paraboloid from the vertex is two-thirds of the height of the paraboloid.

(6) *Centre of gravity and mass of a rod whose density varies as the distance from an extremity.*

Let AB be the rod, MN a small portion of it, then the density at $M \infty AM$.
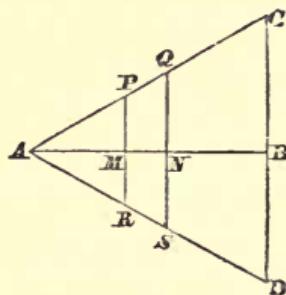
Construct on AB as axis an isosceles triangle CAD, whose base is CD, and draw PMR, QNS parallel to CD; then PR, QS, CD are proportional to the densities at M, N and B; therefore the mass of MN is proportional to a rectangle intermediate to the rectangles PR, MN and QS, MN, which are ultimately in a ratio of equality.

Hence the mass of MN is ultimately proportional to the mass of the rectangle PR, MN, supposed of uniform density, and the moment of MN, with respect to the line CD, is proportional to the moment of the same rectangle, since their distance is the same; hence, by the Lemma, the moment of the whole rod

: the moment of the triangle with respect to CD

:: the mass of the rod : the mass of the triangle;

therefore, the distances of the centres of gravity of the rod and triangle from CD being the same, the centre of gravity of the rod is at a distance $\frac{1}{3}AB$ from B.

Also, the mass of MN being proportional to the area PRN, the mass of the rod is proportional to the area of the triangle ACD, and the mass of a rod of uniform density equal to that at B, and of length AB, being in the same proportion to the rectangle AB, CD, is therefore double of the mass of the rod.

(7) *Centre of gravity of a circular arc.*

Let O be the centre of an uniform circular arc ABC, OB the bisecting radius, aBc a tangent at B, OD parallel to ac, and Aa, Cc parallel to OB.

Let QR be the side of a regular polygon described about the
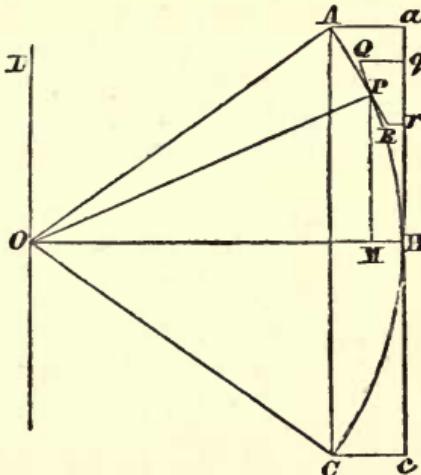
arc, P the point of contact, Qq, Rr perpendicular to ac, and PM to OB. Then, since OP, OB are perpendicular to QR, qr,

$$qr : QR :: OM : OP :: OM : OB;$$

but, since OM, OB are the distances of the centres of gravity of QR and qr from OD, and $QR.OM = qr.OB$, the moments of QR and qr with respect to OD are in a ratio of equality, and the same is true of every side of the circumscribing polygon; therefore, by Cor., Lemma IV., the moment of the arc, which is ultimately that of the polygon, is equal to the moment of ac

$$= ac.OB = \text{chord } AC.\text{radius } OB.$$

Hence, the distance of the centre of gravity of the arc from O

$$= \frac{\text{radius} \times \text{chord}}{\text{arc}}.$$

(8) *Surface of a segment of a sphere.*

Let AKH be the portion of a circle which generates by revolution round AH the spherical segment, O the centre of the circle, PQ the chord of a small arc, PM, QN perpendicular to AH.

Let $AOCD$ be the rectangle circumscribing the quadrant and generating the circumscribing cylinder.

Produce MP, NQ, HK to meet CD in p, q, k. Since PQ is in its limiting position a tangent at P, PQ is ultimately perpendicular to the radius OP, also pq is perpendicular to MP;

$$\therefore PQ : pq :: OP : PM \text{ ultimately,}$$

and the surface generated by PQ is ultimately $2\pi PM.PQ$, Art. 18, $= 2\pi. OP.pq =$ the surface generated by pq.
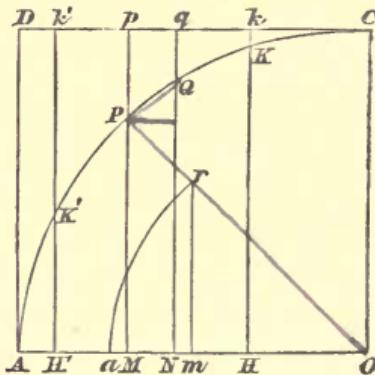
The same is true for each side of the inscribed polygon when the number is indefinitely increased.

Hence the surface generated by AK, or the surface of the spherical segment, is equal to the surface of the circumscribed cylinder cut off by the plane of the base of the segment.

COR. Hence, also, the surface of any belt of a sphere cut off by two parallel planes is equal to the corresponding belt of the cylindrical surface.

(9) *Centre of gravity of a belt of the surface of a sphere contained between parallel planes.*

The moment of the belt generated by PQ with respect to the plane through A, perpendicular to AH, is evidently ultimately equal to that of the belt generated by pq; therefore the moment of any belt generated by $K'K$ is equal to that of the corresponding belt generated by $k'k$.

Hence, the centres of gravity of the two belts are coincident, viz. in the bisection of HH', that is, the distance of the centre of gravity of a spherical belt, contained between parallel planes, is half-way between the two planes.

(10) *Volume of a spherical sector.*

Let the spherical sector be generated by the revolution of the sector AOP about AO.

The volume of the spherical sector is equal to the limit of the sum of a series of pyramids whose vertices are in O, and the sum of whose bases is ultimately the area of the surface of the segment; also the volume of each pyramid is $\frac{1}{3}$ base $\times$ altitude.

Hence, the volume of the spherical sector is one-third of the area of the surface of the spherical segment $\times$ radius

$$= \tfrac{1}{3} . 2\pi AD . Dp . AO = \tfrac{2}{3}\pi AM . AO^{2} = \tfrac{2}{3}\pi AO^{3} \text{ vers} POA.$$

(11) *Centre of gravity of a spherical sector.*

If we suppose each of the pyramids on equal bases, they may be supposed collected at their centres of gravity, whose distances are $\frac{3}{4}AO$ from O ultimately, and they form a mass which may be distributed uniformly over the surface of a spherical segment whose radius is $\frac{3}{4}AO$, viz. that generated by ar, whose centre of gravity will be in the bisection of am, if rm be perpendicular to AH.

Therefore the distance of the centre of gravity of the spherical sector from $O = \frac{1}{2}(Oa + Om) = \frac{3}{4}OA . \cos^{2}\frac{1}{2}POA$.

If the angle POA become a right angle, the distance of the centre of gravity of the corresponding sector, which in this case will become the hemisphere, will be $\frac{3}{8}OA$, as in page 29.

(12) *To find the direction and magnitude of the resultant attraction of a uniform rod upon a particle, every particle of the*

*rod being supposed to attract with a force which varies inversely
as the square of its distance from the attracted particle.*
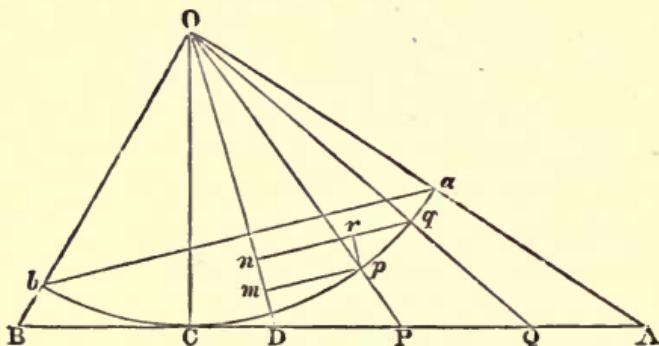
Let AB be the attracting rod, O the particle attracted by the
rod; draw OC perpendicular to AB, join OA, OB, and let a
circle be described with centre O and radius OC meeting OA,
OB in a, b. Let OpP, OqQ be drawn cutting off the small
portions pq, PQ from the arc aCb and the rod, respectively,
and draw PR perpendicular to OQ.

Then $PR : PQ :: OC : OP$ ultimately,

and $pq : PR :: Op : OP$ ....;

$\therefore pq : PQ :: Op^2 : OP^2$ .... ,

and, if aCb be of the same density as the rod and attract
according to the same law,

attraction of pq on O : attraction of PQ :: $\dfrac{pq}{Op^2} : \dfrac{PQ}{OP^2}$ ultimately.

Therefore the portions PQ, pq of the rod and arc attract O
in the same direction with forces which are ultimately equal.

Hence, by Cor., Lemma IV., the resultant attraction of the
rod is the same as that of the arc aCb, which, by symmetry,
is in the direction OD, bisecting the angle AOB.

Again, draw qn perpendicular to OD, pr to qn; then, by
similar triangles, pqr, qOn,

$$pq : qr :: Oq : On;$$

$$\therefore \frac{pq}{Oq^2} \cdot \frac{On}{Oq} = \frac{qr}{OC^2};$$

that is, the resultant attraction of pq in the direction OD is the

same as that of qr at the distance OC; hence the whole resultant attraction of AB is

$$\frac{\mu.ab}{OC^2}, \text{ or } \frac{2\mu}{OC}\sin\frac{1}{2}AOB,$$

where μ is the attraction of a unit of mass at the unit distance.

V.

1. Shew that the area of the sector of an ellipse contained between the curve and two central distances varies as the angle of the corresponding sector of the auxiliary circle.

2. Prove that the volumes of two pyramids will be equal if they stand on the same base, and have their vertices in the same plane parallel to the base.

3. Find the volume of a paraboloid by comparison with the area of a triangle whose vertex and base are those of the generating parabola.

4. Find the centre of gravity of the paraboloid by reference to the same triangle.

5. Find the mass of a straight rod, whose density varies as the square of the distance from one extremity, by comparison with a cone whose axis is the rod.

6. Shew that the orthogonal projection of any plane area on another plane is the given area × the cosine of the inclination of the two planes.
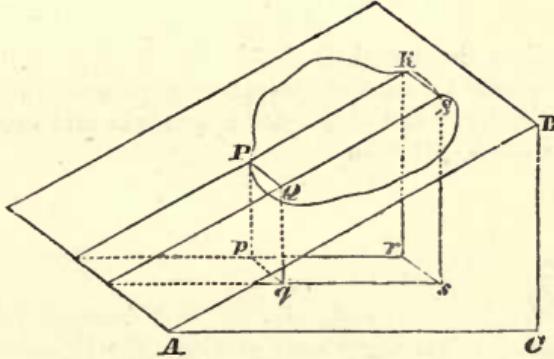
As a first step, prove that, $pqsr$ being the projection of the inscribed parallelogram $PQSR$, $pqsr : PQSR :: \cos BAC : 1$.

7. Find the volume of a hemisphere by comparing the volumes generated by the quadrantal sector and the portion of the circumscribing square which is the difference between the square and the quadrantal sector.

G

VI.

1. Find the volume of a paraboloid generated by the revolution of a semi-cubical parabola, in which $PM^2 \propto AM^3$, by means of a cone on the same axis.

2. Assuming that the area of a belt of a sphere cut off by two parallel planes varies as the perpendicular distance between them, find by the aid of Lemma IV. the area of any portion of the curve of sines.

3. Prove that, if PQ be a small arc of an ellipse, and CD be conjugate to CP, the limit of the sum of all the ratios $PQ : CD$, taken over the whole perimeter of the ellipse, will be 2π.

4. P is any point of a curve OP; OX, OY any lines drawn at right angles through O, PM, PN perpendicular to OX, OY respectively. Prove that, if area OPM : area OPN : : m : 1 always, and the whole system revolve about OX, volumes generated by OPM, OPN will be as m : 2.

5. Prove that the surface generated by the revolution of a semi-circle round its bounding diameter is to the curved surface generated by the revolution of the same semi-circle round the tangent at the extremity of the diameter in the ratio of the length of the diameter to the length of the arc of the semi-circle.

6. Common ordinates MPP', NQQ' are drawn to two ellipses which have a common minor axis, and the outer of which touches the directrices of the inner; shew that the area of the surface generated by the revolution of PQ about the major axis bears a constant ratio to the area $MP'Q'N$.

7. Prove that the area included between an hyperbola and the tangents at the vertices of the conjugate hyperbola is equal to the area included between the conjugate hyperbola and the tangents at the vertices of the hyperbola.

LEMMA V.

All the homologous sides of similar figures are proportional,
whether curvilinear or rectilinear, and their areas are in
the duplicate ratio of the homologous sides.

[Similar curvilinear figures are figures whose curved
boundaries are curvilinear limits of corresponding
portions of similar polygons.

Let $SABCD...$, $sabcd...$ be two similar polygons, of
which SA, AB, BC, ... are homologous to sa, ab,
bc, ... respectively.
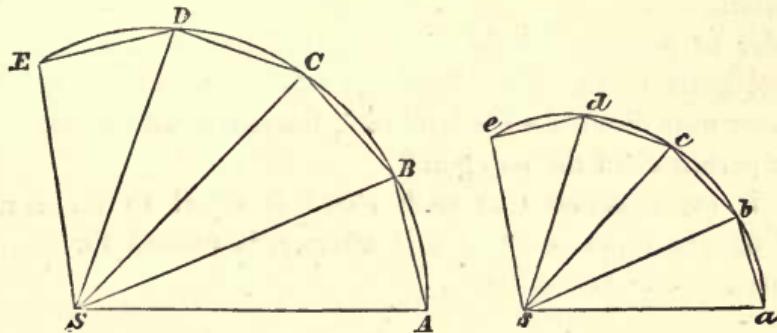
Then $AB : ab :: SA : sa$,

similarly, $BC : bc :: AB : ab :: SA : sa$,

$CD : cd :: BC : bc :: SA : sa$,

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

therefore, componendo,

$AB + BC + CD + ... : ab + bc + cd + ... :: SA : sa.$

Now this, being true for all similar polygons, will be
true in the limit, when the number of the sides AB,
BC, ... and ab, bc, ... is increased, and their lengths
diminished indefinitely; if, therefore, AE, ae be
curves which pass through the angular points A, B, ...
and a, b, ... of the polygons, these curves will be
curvilinear limits of $AB + BC + ...$ and $ab + bc + ...$,

and will be the boundaries of similar curvilinear figures; therefore the curved line AE : the curved line ae

$$:: SA : sa :: SE : se.$$

Again, polygon $SABC...$: polygon $sabc...$ $:: SA^2 : sa^2$, and this is true in the limit; hence, by Lemma III. Cor. 2, curvilinear area SAE : curvilinear area sae

$$:: SA^2 : sa^2 :: AE^2 : ae^2 :: SE^2 : se^2.$$

Q. E. D.]

Observations on the Lemma.

24. In order to deduce the properties of similar curves, it is premised, as before mentioned under Cor. 4, Lemma III., that, if a *finite* portion of a curve be taken, and if a polygon be inscribed in the curve, the sides of which are chords taken in order of portions of the curve, and the number of sides of the polygons be increased indefinitely, and the magnitudes a the same time diminished indefinitely, the curve will be the limit of the perimeter of the polygon.*

It is *not* assumed that each chord is equal to the corresponding arc ultimately; this is afterwards proved for a continuous curve in Lemma VII.

Criteria of Similarity.

25. From the *definition* of similar curve lines, that they are curvilinear limits of homologous portions of similar polygons, the following criteria of similarity can be deduced, all of which are very convenient in practice; namely:

(1) One curve line is similar to another when, if *any* polygon be inscribed in one, a similar polygon can be inscribed in the other.

(2) If two curves be similar, and any point S be taken in the plane of one curve, another point s can be found in the plane of the other, such that, *any* radii SP, SQ being drawn in the first, radii sp, sq can be drawn in the second, inclined at

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* Whewell's *Doctrine of Limits.*

the same angle as the former, and such that the following proportion will hold,

$$sp : sq :: SP : SQ.$$

(3) If two curves be similar, and in the plane of one curve *any* two lines OX, OY be drawn, two other lines ox, oy can be drawn in the plane of the other curve, inclined at the same angle, having the property that the abscissa and ordinate OM, MP of any point P in the first being taken, the abscissa and ordinate om, mp of a corresponding point p in the second will be proportional to the former, viz.,

$$om : mp :: OM : MP.$$

And the converse propositions can also be deduced, that if these proportions hold, the curves will be similar.

26. In order to illustrate test (1), let the arcs AB, ab of two circles have the same centre C, and let the bounding radii be coincident in direction.
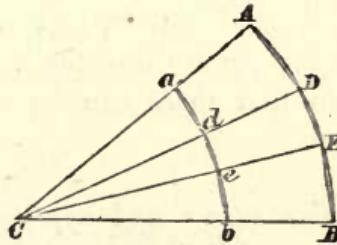
Let $ADEB$ be *any* polygon inscribed in AB, and let CD, CE cut ab in d, e; join ad, de, eb, these are parallel to AD, DE, EB respectively, and $ad : de : eb :: AD : DE : EB$; hence $adeb$ is similar to $ADEB$; and therefore the arcs ab, AB are similar.

27. Test (2) may be deduced as follows:

If $ABCD...$, $abcd...$, fig. p. 43, be corresponding portions of similar polygons, AB, BC, ... ab, bc, ... being homologous sides, and AS, BS, ... be drawn to any point S, construct the triangle sab equiangular with SAB, and join sc, sd, ...

Then $sb : SB :: ab : AB :: bc : BC$, and $\angle SBC = \angle sbc$;

therefore SBC, sbc are similar triangles;

hence $sc : SC :: sb : SB :: sa : SA$;

and similarly for sd, se, &c.

Hence, if two polygons be similar, and any point be taken in one, another point can be found in the other, such that the radii drawn to corresponding angular points will be proportional and include the same angles.

If we now increase the number of sides indefinitely and diminish their magnitude, the same property will hold with respect to the curvilinear limit of the polygon.

Test (3) can be deduced from test (1) in a similar manner.

Centres of Similitude.

28. When two similar curves are so situated that a point can be found, such that the radii drawn from that point, either in the same or opposite directions, are in a constant ratio, such a point is called a *centre of similitude*.

If the radii be measured in the same direction, the point will be a centre of *direct* similitude, and of *inverse* similitude if they be measured in opposite directions.

It is easily shewn that there can be only one centre of similitude of one kind.

Properties of similar curves and application of tests of Similarity.

(1) *Similar conterminous arcs, which have their chords coincident, have a common tangent.*
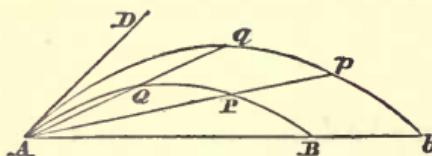
Let APB, Apb be similar conterminous arcs, ABb the line of their chords, AQq, APp any straight lines meeting the curves in Q, q and P, p respectively; then A will evidently be a centre of direct similitude for the two curves; therefore $AQ : Aq :: AP : Ap$; hence AP, Ap are similar portions of

the curves, and arc AP : arc Ap :: AP : Ap :: AB : Ab; therefore the arcs AP, Ap vanish simultaneously, or, when AP assumes its limiting position AD for the curve APB, this is also the limiting position of Ap for the curve Apb, that is, the curves have a common tangent.

(2) *To find the centres of direct and inverse similitude of any two circles.*
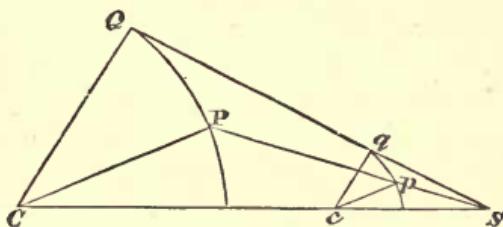
If one of the circles do not lie entirely within the other, let S be the intersection of two common tangents to the circles which intersect in the *produced* line Cc joining their centres, and let CQ, cq be radii to the points of contact.

Draw SpP through S cutting the circles in p, P, then cq is parallel to CQ, and $CP : cp :: CQ : cq :: CS : cS$;

$$\therefore \ CS : CP :: cS : cp;$$

also CPS, cpS are each greater or each less than a right angle, and CSP is common to the triagles CPS, cpS; therefore the triangles are similar, Euclid VI. 7, and the sides about the angle CSP are proportional, that is, $SP : Sp :: SC : Sc$; therefore S is the centre of direct similitude.

Similarly, the intersection of two common tangents which cross between two circles is the centre of inverse similitude.

(3) *To find the condition of similarity of two conic sections.*

Let the conic sections be placed so that their directrices
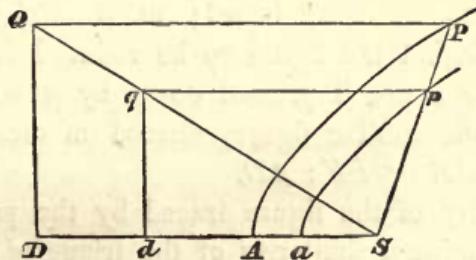
are parallel and foci coincident, and let SpP be any line
through the focus meeting them in p, P; draw $SaAD$ and
PQ perpendicular to the directrix DQ of AP, and join SQ,
and let pq, parallel to PQ, meet it in q, and draw qd per-
pendicular to SD.

Then $Sd : SD :: Sq : SQ :: Sp : SP$; and, if the curves
be similar, $Sp : SP$ will be a constant ratio; therefore $Sd : SD$
is a constant ratio, and dq is a fixed straight line for all
positions of p; also, since $pq : Sp :: PQ : SP$, $pq : Sp$ is a
constant ratio; therefore qd is the directrix of ap, and, the
constant ratio being the same in both, the eccentricities are
the same.

(4) Instruments, like the *Pantagraph* and the *Eidograph*, for
copying plans on an enlarged or reduced scale are founded upon
the properties of similar figures; as are also other methods of
copying, such as by dividing plans or pictures into squares.

The Pantagraph is an instrument for drawing a figure
similar to a given figure on a smaller or larger scale; one of
its forms is as in the figure. AD, EF, GC and AE, DG, FC
are two sets of parallel bars, joined at all the angles by
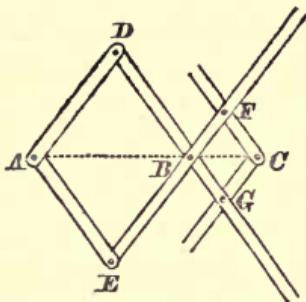
compass-joints; at B is a point, which serves to fix the
instrument to the drawing board; at A is a point which is
made to pass round the figure to be reduced or enlarged; at
C is a hole for a pencil pressed down by a weight, and the
pencil traces the similar figure, altered in dimensions in the
ratio of $BC : AB$ or $BF : AD$.

The similarity of the figure traced by the pencil is a con-
sequence of continual similarity of the triangles ABD, BFC.

By changing the positions of the pegs at F and G the figure described by C may be made of the required dimensions.

For a description of the Eidograph, invented by Professor Wallace, see the *Transactions of the Royal Society of Edinburgl*, vol. XIII.

(5) *Volume of a cone whose base is a plane closed figure of any form.*

Let V be the vertex, AB the base, VH perpendicular to the base from V; let VH be divided into n equal portions, of
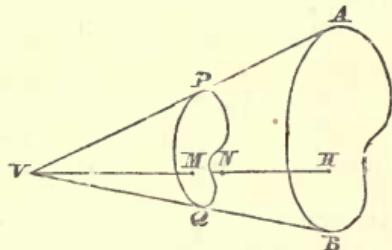
which MN is the $(r+1)^{\text{th}}$; and let PQ be the section through M parallel to AB.

Take VPA *any* generating line of the cone meeting the section PQ and the base AB in PA respectively, then

$$PM : AH :: VM : VH;$$

therefore PQ is similar to AB, M, H being similarly situated points; and, by Lemma V.,

$$\text{area } PQ : \text{area } AB :: r^2 : n^2,$$
$$\text{also } MN : VH :: 1 : n;$$

therefore the volume of the cylinder whose base is PQ and height $MN = \dfrac{r^2}{n^3} \times \text{area } AB \cdot VH$, and the volume of the cone, by Lemma II., is one-third of the cylinder whose base is AB and height VH.

VII.

1. Apply a criterion of similarity to shew that segments of circles which contain equal angles are similar.

2. From the definition of an ellipse, as the locus of a point the sum of whose distances from two fixed points is constant, shew that ellipses are similar when the eccentricities are equal.

H

3. Prove that the centre of an ellipse is a centre of inverse similitude of two opposite equal portions of the circumference of the ellipse.

4. Employ the properties of similar figures to inscribe a square in a given semicircle.

5. Construct, by means of similar figures, two circles, each of which shall touch two given straight lines and pass through a given point.

6. Deduce the position of the centre of gravity of a circular sector from that of a circular arc; shew that the distance from the centre is $\dfrac{2}{3} \cdot \dfrac{\text{radius} \times \text{chord}}{\text{arc}}$.

7. If A be the vertex of a conical surface, G the centre of gravity of the base, H that of the volume of the conical figure, shew that $AH = \frac{3}{4}AG$.

8. Find the centre of gravity of the surface of a right cone on a circular base. Does the method apply to the surface of an oblique cone?

LEMMA VI.

If any arc ACB given in position be subtended by a chord AB, and if at any point A, in the middle of continuous curvature, it be touched by the straight line AD produced in both directions, then, if the points A, B approach one another and ultimately coincide, the angle BAD contained by the chord and tangent will diminish indefinitely and ultimately vanish.

For, if that angle do not vanish, the arc *ACB* will contain with the tangent *AD* an angle equal to a
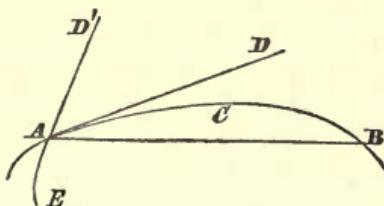
rectilineal angle, and therefore the curvature at the point *A* will not be continuous, which is contrary to the hypothesis, that *A* was in the middle of continuous curvature.

Definitions of a Tangent to a Curve.

29. (1) If a straight line meet a curve in two points *A*, *B*, and if *B* move up to *A*, and ultimately coincide with *A*, *AB* in its limiting position will be a tangent to the curve at the point *A*.

If two portions of a curve *EA* and *AB* cut one another at a finite angle in *A*, there will be two tangents *AD*, *AD'*, which will be the limiting positions of straight lines *AB* and *AE*, when *B* and *E* move up to *A* along the different portions *EA* and *BA* of the curve respectively. And, similarly, if there be a multiple point in *A*, in which several branches of the curve cut one another at finite angles.

(2) The tangent is the direction of the side of the polygon, of which the curve is the curvilinear limit, when the number of sides are increased indefinitely.

This is founded on the same idea of a tangent as definition (1).

(3) The tangent to a curve at any point is the direction of the curve at that point.

In order to apply geometrical reasoning to the tangent by employing this definition, we are obliged to explain the notion of the direction of a curve, by taking two points very near to one another, and asserting that the direction of the curve is the limiting position of the line joining these points when the distance becomes indefinitely small, a statement which reduces this definition to the preceding.

Observations on the Lemma.

30. "Curvatura Continua," if we consider curves as the curvilinear limits of polygons, requires the curves to be limits of polygons whose angles continually increase as the number of the sides increase, and may be made to differ from two right angles by less than any assignable angle before the assumption of the ultimate form of the hypothesis.

If, however, as we increase the number of sides and diminish their magnitude, one of the angles remains less than two right angles by any finite difference, the curvature of the curvilinear limit is discontinuous, and the form is that of a pointed arch, in which the two portions cut one another at a finite angle.

A curve may be of continued curvature for one portion between two points, while for another its curvature changes "per saltum."

Thus, if ABC be a curve forming at B a pointed arch, it
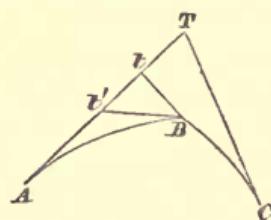
may be of continued curvature from B to A and from C to B, though not from C to A.

In this case the tangents in passing from C to A assume all

positions intermediate to CT, Bt, and Bt', TA, but at B they pass from Bt to Bt' without assuming the intermediate positions.

31. "In medio curvaturæ continuæ," implies that the point A in the enunciation of the Lemma is not such a point as B in the last figure, but that, in passing from a point on one side of A to another on the other side, the tangents pass through all the intermediate positions.

The curvature is supposed to be in the same direction in the figure of the Lemma, which in all curves of continuous curvature is possible, if B be taken sufficiently near to A at the commencement of the change in the construction.

If the point A be not "in medio curvaturæ continuæ," two tangents AD, AD' may be drawn at A to the two parts of the curve, and the curve BCA will make a finite angle with one of the tangents AD'.

But, even in this case, the angle between the chord and that tangent which belongs to the portion of the curve considered continually diminishes and ultimately vanishes.

The Subtangent.

32. DEF. The part of the line of abscissæ intercepted between the tangent at any point and the foot of the ordinate of that point is called the *subtangent*.

33. The subtangent may be employed as follows, to find a tangent at any point of a curve.

Let OM, MP be the abscissa and ordinate of a point P in
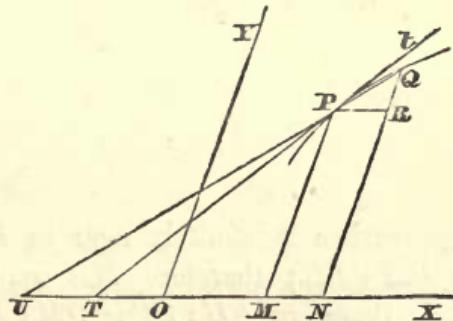
a curve, and let Q be a point near P, ON, NQ its abscissa and ordinate.

Let QPU meet OX the line of abscissæ in U; then, if PR parallel to OM meet QN in R,

$$PM : MU :: QR : PR :: QN - PM : ON - OM.$$

Now as Q approaches to P, the limiting position of QPU is that of the tangent at P, viz. tPT, and $PM : MT$ is the limiting ratio of $QN - PM : ON - OM$.

The Polar Subtangent and the Inclination of the Tangent to the Radius Vector, at any Point of a Spiral.

34. DEF. Let S be the pole, PT the tangent to the curve at any point P, and let ST, perpendicular to SP, meet PT in T; then ST is called the *polar subtangent* at the point P.

35. *To find the inclination of the tangent at any point of a curve to the radius vector.*

Let Q be a point near P, QM perpendicular to SP, produced if necessary, QR the circular arc, centre S, meeting SP in R.

Let QP meet ST in U, then

$$SU : SP :: QM : PM,$$
$$\text{and } MR : QM :: QM : SM + SR,$$
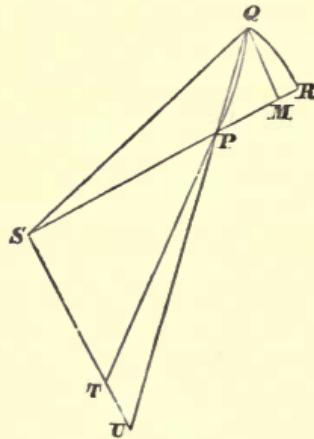
but, when Q approaches indefinitely near to P, QM vanishes compared with $SM + SR$; therefore MR vanishes compared with QM or PM; therefore $SU : SP :: QM : PR$, ultimately; therefore $ST : SP$ is the limiting ratio of $QR : PR$; or $QR : SQ \sim SP$.

Hence ST, and also the trigonometrical tangent of the angle SPT between the tangent and the radius vector can be found.

Illustrations.

(1) *If SY be the perpendicular on the tangent PY at P in a curve, Y will trace out a curve, called the pedal of the original curve; to shew that if YZ be a tangent to the locus of Y, SZ perpendicular to it, $SY^2 = SP . SZ$.*

Let P' be a point near P, SY' perpendicular on $P'P$, SZ perpendicular on $Y'Y$.

Since angles SYP, $SY'P$ are right angles, a semicircle on SP will pass through Y, Y'; therefore the angles $SY'Y$, SPY in the same segment will be equal; the right angles SZY', SYP also are equal; therefore the triangles SPY, $SY'Z$ are similar, and $SZ : SY' :: SY : SP$; but, ultimately, as P' moves
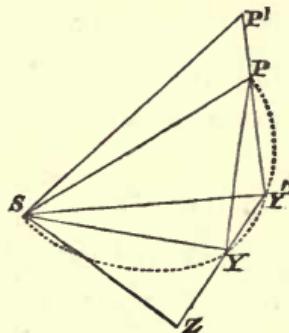
up to P, $P'PY'$ becomes the tangent at P, and $Y'YZ$ that at Y to its locus, also $SY' = SY$;

$$\therefore SZ . SP = SY^2.$$

(2) *To find the subtangent in the semi-cubical parabola.*

In the semi-cubical parabola $PM^2 \propto OM^3$;

$$\therefore QN^2 - PM^2 : PM^2 :: ON^3 - OM^3 : OM^3,$$

but $QN + PM = 2PM$,

and $ON^2 + ON . OM + OM^2 = 3OM^2$, ultimately;

$$\therefore QN - PM : \tfrac{1}{2}PM :: ON - OM : \tfrac{1}{3}OM \text{ ultimately,}$$

and $QN - PM : PM :: ON - OM = MT$;

therefore MT is two-thirds of OM.

(3) *To find the inclination of the tangent at any point of a cardioid to the radius vector.*

DEF. If $BqpC$ be a circle, whose centre is S and diameter BC, and pm be drawn perpendicular to BC; then, if Sp be produced to P, making $SP = Bm$, P will trace out a *cardioid APS*.
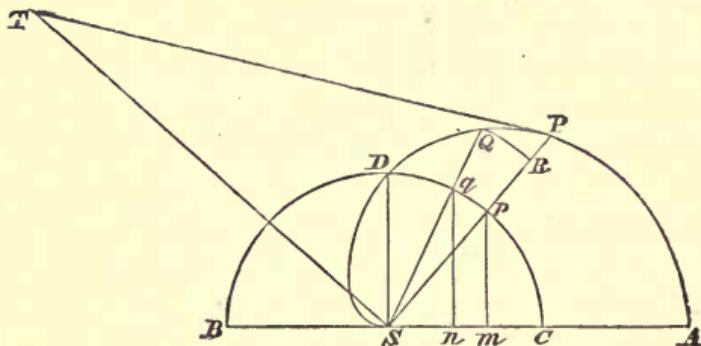
Making the same construction as before, in Art. 35,

$$ST : SP :: QR : SP - SQ \text{ ultimately.}$$

Let SQ meet the circle in q, and draw qn perpendicular to BC,

$$\text{then } QR : pq :: SP : Sp \text{ ultimately,}$$
$$\text{also } pq : mn :: Sp : pm \dots\dots\dots,$$
$$\therefore QR : mn :: SP : pm \dots\dots\dots;$$
$$\text{but } mn = Bm - Bn = SP - SQ;$$
$$\therefore QR : SP - SQ :: SP : pm \text{ ultimately;}$$
$$\therefore ST : SP :: Bm : pm;$$
$$\text{hence } \angle PTS = \angle pBm = \tfrac{1}{2} \angle PSA;$$

and it follows that the cardioid cuts the axis SCA at right angles, that it touches SB at S, and that it cuts the circle BDC at an angle equal to half a right angle.

LEMMA VII.

If any arc, given in position, be subtended by the chord AB, and at the point A, in the middle of continuous curvature, a tangent AD be drawn, and the subtense BD, then, when B approaches to A and ultimately coincides with it, the ulimate ratio of the arc, the chord, and the tangent to one another, is a ratio of equality.

For whilst the point B approaches to the point A, let AB, AD be supposed always to be produced to points b and d at a finite distance, and let bd be drawn parallel to the subtense BD, and let the arc Acb be always similar to the arc ACB, and have, therefore, ADd for its tangent at A.
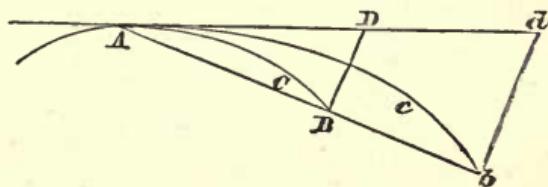
But, when the points B, A coincide, the angle bAd, by the preceding Lemma, will vanish, and therefore the straight lines Ab, Ad, which are always finite, and the arc Acb, which lies between them [and is of continuous curvature in one direction, if the change commence when B is near enough to A], will coincide ultimately, and therefore will be equal.

Hence, also, the straight lines AB, AD and the intermediate arc ACB, which are always proportional to them, will vanish together, and have an ultimate ratio of equality to one another.

Cor. 1. Hence if BF be drawn through B parallel to the tangent, always cutting any straight line AF passing through A in F, then BF will have ultimately to the vanishing arc ACB a ratio of equality,

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since, if the parallelogram $AFBD$ be completed, it
will always have a ratio of equality to AD.
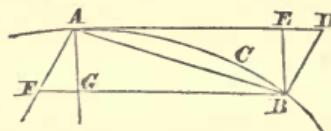
Cor. 2. And if through B and A be drawn many
straight lines BE, BD, AF, AG cutting the tangent
AD and BF, parallel to it; the ultimate ratios of all
the abscissæ AD, AE, BF, BG and of the chord and
arc AB to one another will be ratios of equality.

Cor. 3. And, therefore, all these lines in every argu-
ment concerning ultimate ratios may be used indif-
ferently one for the other.

Observations on the Lemma.

36. Def. The *subtense* of the *angle of contact* of an arc is a
straight line drawn from one extremity of the arc to meet, at
a finite angle, the tangent to the arc at the other extremity.

This subtense is the secant which defines the limited line
called, in the Lemma, " the tangent."

The chord is called by Newton the *subtense* of the *arc*, see
Lemma XI.

37. In the construction for this Lemma, BD must be a
subtense, *i.e.* inclined throughout the change of position at a
finite angle to the tangent, for, otherwise, the angles BAD
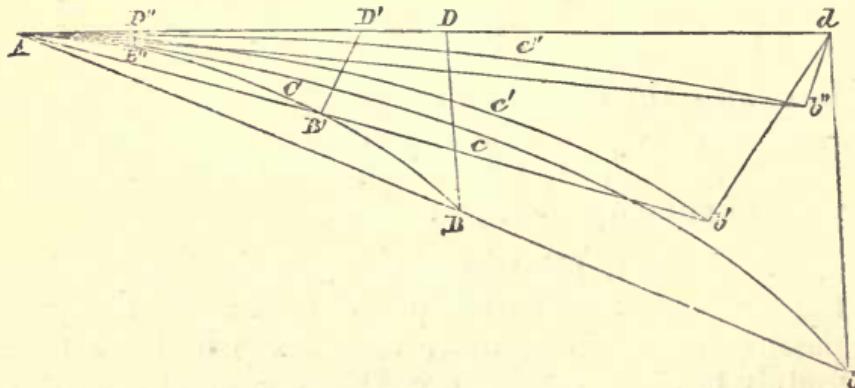
and ADB being then both small, the ultimate ratio of the chord to the tangent might be any finite ratio instead of being one of equality.

This is the only limitation of the motion of BD; the figure representing changes which may take place in the approach towards the ultimate state of the hypothesis.

Here b, d are the distant points, that is, points at a finite distance from A; BD, $B'D'$, $B''D''$ are consecutive positions of the subtense, when B approaches towards A, and db, db', db'' are parallel to these, $Ac'b'$, $Ac''b''$ are the forms of Acb *changed* so as to be always similar to the corresponding portion of ACB cut off by the chord.

It should be remarked that the curve Acb is not intermediate in *magnitude* to the two lines Ab, Ad, but only in *position;* for example, Ab may be equal to Ad, if BD make equal angles with the two lines, and the curve line will then be greater than either Ab or Ad; but it becomes in all cases less bent, until it is ultimately rectilinear; hence the three Acb, Ab, Ad will be ultimately equal, the only alternative being that the curve may become doubled up, as in the figure,
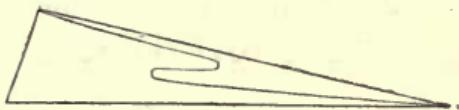
which is precluded by the supposition that the curvature near A is continued in the same direction throughout the passage from B to A.

38. *The subtense ultimately vanishes compared with the arc.*

For $BD : ACB :: bd : Acb$, and, since bd vanishes and Acb remains finite in the limit, the ratio $BD : ACB$ ultimately vanishes. It will be afterwards seen that in curves of finite curvature BD varies as the square of ACB ultimately.

The ultimate equality of the lines AD, AE with the chord or arc, whatever be the direction of the subtense, is due to the vanishing of BD, and therefore of DE with respect to AD.

39. *If two curves of continuous curvature which do not intersect have a common chord, the length of the exterior curve will be*

*greater than that of the interior, provided that the curvature of
the interior be always in the same direction.*

Let $AcdeB$, $ACDEFB$ any two polygons, having a common
side AB, be such that the first lies entirely within the second
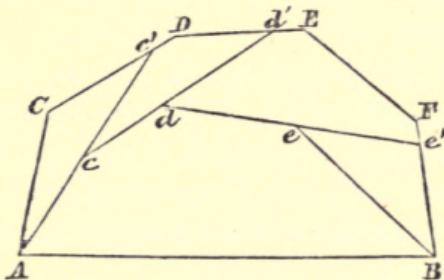
and that neither has internal angles, the perimeter of the first
is less than that of the second.

For, produce Ac, cd, de to meet the perimeter of the exterior
in c', d', e'; then $AC + Cc' > Ac'$; $\therefore ACDEFB > Ac'DEFB$;

$$\text{similarly } Ac'DEFB > Acd'EFB, \text{ and on on;}$$

$$\therefore a \text{ fortiori, } ACDEFB > AcdeB.$$

And, since the same is true in the limit, when the number
of sides is increased indefinitely, the curvilinear limits of the
polygons have the same property, and the proposition is proved.

LEMMA VIII.

If two straight lines AR, BR make with the arc ACB, the chord AB, and the tangent AD, the three triangles RACB, RAB and RAD, and the points A, B approach one another; then the ultimate form of the vanishing triangles is one of similitude, and the ultimate ratio one of equality.

For, whilst the point B is approaching the point A, let AB, AD, AR be always produced to points b, d, r at a finite distance, and rbd be always drawn parallel to RD, and let the arc Acb be always similar to the arc ACB, and therefore have Dd for the tangent at A.
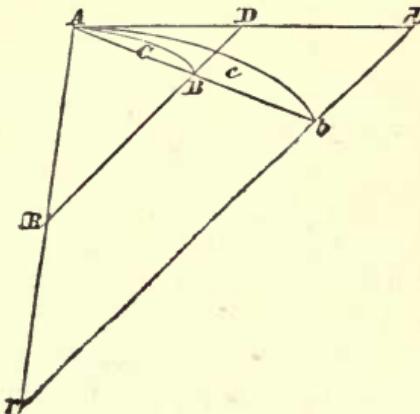
Then, when the points B, A coincide, the angle bAd will vanish, and therefore the three triangles rAb, $rAcb$, rAd will coincide, and will therefore in that case be similar and equal. Hence also RAB, $RACB$, RAD, which are always similar and proportional to these, will be ultimately similar and equal to one another.

Cor. And hence, in every argument concerning ultimate ratios, these triangles can be used indifferently for one another.

Observations on the Lemma.

40. If RB throughout the change in the hypothesis make a finite angle with RA, the three triangles rAb, $rAcb$, rAd will

remain always finite, and will be ultimately identical and equal. But, if the angle ARB be ultimately not finite, for example, if RB revolve round a fixed point R, the three triangles rAB, ... will become infinite, since r will move to r' and so on to an infinite distance, and there will be the same kind of objection
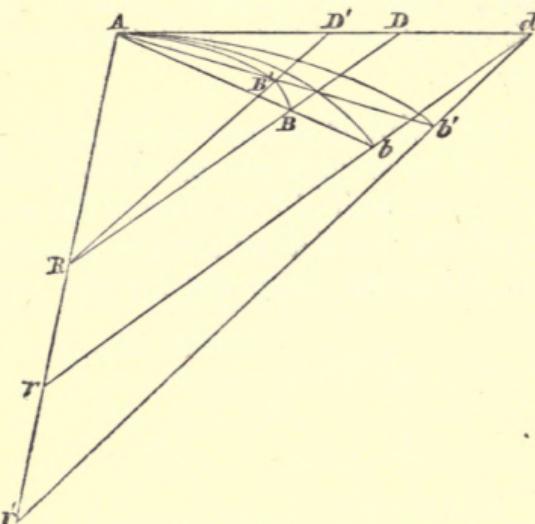
to dealing with these infinite triangles, as to reasoning immediately upon the relation of the triangles RAB, RAD in the former case.

In this case we can at once deduce the equality of the triangles without producing AD to a point d at a finite distance. For, the ratio of the difference of RAD and RAB to RAB is $BD : RB$, which vanishes ultimately, since RB is finite in this case; hence RAB and RAD and also the curvilinear triangle, which is intermediate in magnitude to them, will be ultimately in a ratio of equality.

LEMMA IX.

If a straight line AE and curve ABC, given in position, cut one another in a finite angle A, and ordinates BD, CE bs drawn, inclined at another finite angle to that straight line, and meeting the curve in B, C ; then, if the pointe B, C move up together to the point A, the areas of the curvilinear triangles ABD, ACE will be ultimately to one another in the duplicate ratio of the sides.

For, as the points *B*, *C* are approaching the point *A*, let *AD*, *AE* be always produced to the points *d*, *e* at a finite distance, such that *Ad* : *Ae* :: *AD* : *AE*; and
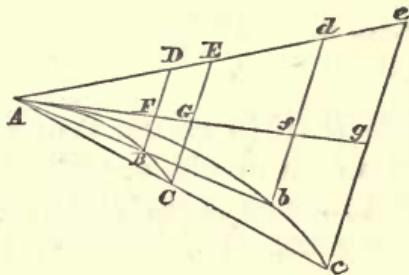
let the ordinates *db*, *ec* be drawn parallel to *DB*, *EC* meeting the chords *AB*, *AC* produced in *b*, *c*.

Then [since *Ab* : *AB* :: *Ad* : *AD* :: *Ae* : *AE* :: *Ac* : *AC*, and therefore *Ab* : *Ac* :: *AB* : *AC*] a curve *Abc* can be supposed to be drawn always similar to *ABC*, while *B* and *C* move up to *A*.

Let the straight line *Ag* be drawn touching both curves at *A*, and cutting the ordinates *DB*, *EC*, *db*, *ec* in *F*, *G*, *f*, *g*.

[Now areas *ABD*, *Abd*, by Lemma V., are always in the duplicate ratio of *AD*, *Ad*, and areas *ACE*, *Ace* in the duplicate ratio of *AE*, *Ae*, and *AD* : *Ad* :: *AE* : *Ae*;

therefore *ABD* : *Abd* :: *ACE* : *Ace*,

and *ABD* : *ACE* :: *Abd* : *Ace*.]

If, then, the points *B* and *C* move up to *A* and ultimately coincide with it, the angle *cAg* will ultimately vanish,

and the curvilinear areas Abd, Ace will coincide with the rectilinear triangles Afd, Age, and therefore will be ultimately in the duplicate ratio Ad, Ae.

But ABD, ACE are proportional to Abd, Ace always, also, AD, AE are proportional to Ad, Ae; therefore also areas ABD, ACE are ultimately in the duplicate ratio of AD, AE.

Observations on the Lemma.

41. By a finite angle is to be understood an angle less than two right angles, and neither indefinitely small nor indefinitely near to two right angles.

The angles between AD and the curve and between AD produced and BD are different finite angles, because otherwise BD would not meet the curve.

42. If the angle DAF be greater than a right angle, the figure may assume a form in which AD will lie below ABC; in this case DB, EC, ... must be produced to meet the tangent, and the argument may proceed in the same manner as before.

43. It is not necessary that d and e be fixed, but only that they remain at a finite distance from A, and that the proportion be retained; and the first part of this observation applies to d in the previous Lemmas.

The student, by reference to Arts. 37 and 40, will be able to exhibit the change in the figure which will correspond to a change of the position of B and C in the progress towards the ultimate position.

44. When the angle CAG vanishes, the curvilinear areas Abd, Ace coincide with the rectilinear triangles Afd, Age, and so are in the duplicate ratio of $Ad : Ae$. But if the angle DAF be not finite, those triangles will not themselves be finite, and the object aimed at by producing to a finite distance will not be attained.

The fact is, that the triangle Adb is made up of the triangle Adf and the curvilinear triangle Afb, of which the latter is indefinitely small ultimately, and the former is finite; therefore,

in the Lemma, Afb vanishes compared with Adf; but this will not be so if Adf be indefinitely small, the ratio of the triangles AFB, AGC must, therefore, be found by another process, and it will be found, by referring to Lemma XI., that the ratio will be ultimately that of the cubes of the arcs if the curvature of the curve at A be finite.

VIII.

1. RQq is a common subtense to two curves PQ, Pq, which have a common tangent PR at P. When RQq approaches to P, RQ and Rq ultimately vanish; will the ratio $RQ : Rq$ be ultimately a ratio of equality?

2. If PY, a tangent to an ellipse at P, meet the auxiliary circle in Y, and ST be perpendicular to the tangent at Y, ST will vary inversely as HP.

3. If a subtense BD be drawn to meet the tangent at A at a finite angle a, which remains constant as B moves up to A, and DB meet the normal at A in C, shew that the ultimate ratio of BC to AB will be $\sec a$.

4. In the curve in which the abscissa varies as the cube of the ordinate, shew that the subtangent is three times the abscissa.

5. Prove that the extremity of the polar subtangent from the focus of a conic section is always in a fixed straight line.

6. AB is a diameter of a circle, P a point contiguous to A, and the tangent at P meets BA produced in T; prove that ultimately the difference of BA, BP will be equal to one-half of TA.

7. In any curve, if Q be the intersection of perpendiculars to two consecutive radii vectores through their extremities, and SY be the perpendicular from the pole S on the tangent at P, prove that ultimately $SP^2 = SY . SQ$.

8. PQ, pq are parallel chords of an ellipse whose centre is C; shew that, if p move up to P, the areas CPp, CQq will be ultimately equal.

9. From a point in the circumference of a vertical circle a chord and tangent are drawn, the one terminating at the lowest point, and the other in the vertical diameter produced; compare the velocities acquired by a heavy body in falling down the chord and tangent when they are indefinitely diminished.

10. A point moves so that the product of its distances from two fixed points is constant; shew that the normal to its path divides the angle between the two radii into two whose sines are proportional to the radii.

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IX.

1. On the radii vectores of a curve as diameters circles are described; find their envelope.

2. If the intercept PQ between two curves of their common radius vector OPQ be constant, and the normals at P and Q intersect in N, ON will be at right angles to OPQ.

3. A right angle slides on any oval curve, so that the sides containing the right angle always touch the curve; shew that the angle one tangent makes with the tangent to the locus of the vertex is equal to that which the other tangent makes with the chord of contact.

Hence shew that, if the oval be an ellipse, the locus of the vertex will be a circle concentric with the ellipse.

4. A point moves so that the rectangle, whose sides are equal to the distances of the point from a given point and a given straight line, is equal to the square described on the perpendicular from the given point on the given line. Find the position of the point at which the tangent to the curve passes through the fixed point.

5. Two points A, B describe two curves according to any finite and continuous law. If A', B' be the consecutive positions of A, B, and ABC, $A'B'C'$ be similar triangles, then the corresponding sides of the two triangles will ultimately intersect in the points P, Q, R, such that $\dfrac{AA'.BC}{QR} = \dfrac{BB'.CA}{RP} = \dfrac{CC'.AB}{PQ}$.

6. If $SP^2 = AB.PM$, where PM is perpendicular to a fixed straight line, prove that the locus of the centre of the circle circumscribing the triangle formed by the tangent, the radius vector, and the polar subtangent, will be a straight line.

7. In the figure on page 30 let FB' be taken equal to AB, and let the corresponding ordinate to the curve be $B'E'$; prove that the subtangent at E' varies inversely as that at E.

8. In the hyperbolic spiral, in which the radius vector varies inversely as the spiral angle, prove that the subtangent is constant.

9. In the spiral of Archimedes, in which the radius vector varies directly as the angle, prove that if a circle be described, of which a radius is the radius vector of the spiral, the polar subtangent will be equal to the arc of the circle subtended by the spiral angle.

LEMMA X.

The spaces which a body describes [from rest] under the action of any finite force, whether that force be constant or else continually increase or continually diminish, are in the very beginning of the motion in the duplicate ratio of the times.

[Let the times be represented by lines measured from A, along AK, and the velocities generated at the end of those times by lines drawn perpendicular to AK. Suppose the time represented by AK to be divided into a number of equal intervals, represented by AB,
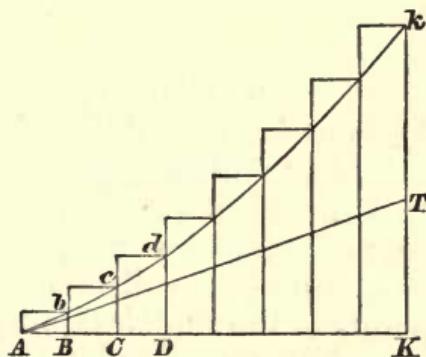
BC, CD, ..., let Bb, Cc, Dd, ...Kk represent the velocities generated in the times AB, AC, ...AK respectively, and let $Abcd$ be the curve line which always passes through the extremities of these ordinates. Complete the parallelograms Ab, Bc, Cd, ...

In the interval of time denoted by CD, the velocity continually changes from that represented by Cc to that represented by Dd, and therefore CD being taken small enough, the space described in that time is intermediate between the spaces represented by the parallelograms Dc and Cd; therefore the spaces described in the times AD, AK are represented by areas which are intermediate between the sums of the parallelograms inscribed in, and circumscribed about, the curvilinear areas ADd, AKk respectively.

Therefore, by Lemma II., the number of intervals being increased, and their magnitudes diminished indefinitely, the spaces described in the times AD, AK are proportional to the curvilinear areas ADk, AKk.

Now the force being finite, the ratio of the velocity to the time is finite; therefore $Kk : AK$ is a finite ratio, however small the time is taken; hence, if AT be the tangent to the curve line at A, meeting Kk in T, $KT : AK$ will be a finite ratio; therefore the angle TAK will be finite, or AK will meet the curve at a finite angle.

Hence, by Lemma IX., if AD, AK be indefinitely diminished, area ADd : area AKk :: AD^2 : AK^2; therefore, in the beginning of the motion, the spaces described are proportional to the squares of the times of describing them.   Q. E. D.]

Cor. 1. And hence it is easily deduced that the errors of bodies describing similar parts of similar figures in proportional times, which are generated by any equal forces acting similarly upon the bodies, and which are measured by the distances of the bodies from those points of the similar figures, to which the same bodies would have arrived in the same proportional times without the action of the disturbing forces, are approximately as the squares of the times in which they are generated.

Cor. 2. But the errors which are generated by proportional forces, acting similarly at similar portions of similar figures, are approximately as the forces and the square of the times conjointly.

Cor 3. The same is to be understood of the spaces which bodies describe under the action of different forces. These are, in the beginning of the motion, conjointly, as the forces and the squares of the times.

Cor. 4. Consequently, in the beginning of the motion the forces are as the spaces described directly, and the squares of the times inversely.

Cor. 5. And the squares of the times are as the spaces described directly and the forces inversely.

The proof given in the original Latin is as follows:

Exponantur tempora per lineas *AD, AE,* et velocitates genitæ per ordinatas *DB, EC*; et spatia, his velocitatibus descripta, erunt ut areæ *ABD, ACE* his ordinatis descriptæ, hoc est, ipso motus initio (per Lemma IX.) in duplicata ratione temporum *AD, AE.*

Q. E. D.

45. This proof has been amplified in order to exhibit in what manner the description of areas, by the flux of the ordinates, corresponds to that of spaces by the velocities represented by the ordinates; also to shew the propriety of the application of the ninth Lemma by reference to the definition of finite force which may be stated as follows: *A force is finite when the ratio of the velocity generated in any time to the time in which it is generated, is finite, however small the time be taken.*

Observations on the Lemma.

46. In the proof of this Lemma, time is represented by the length of a straight line, and a distance traversed by a body is represented by an area.

If the length of a straight line be always proportional to the period of time elapsed, the straight line will be a proper representation of the time. Thus a length of n inches has the same ratio to one inch which an interval of n seconds has to one second; and on this scale the length n inches is a proper representation of n seconds.

If an area be always in the same ratio to the unit of area that the length of a straight line is to the unit of length, the area will be a proper representation of the length of the straight line.

Thus, if *Ab* be one foot, *AB,* n feet, *Ac* one inch, and *AC,* t inches: complete the parallelograms *ABDC, Abdc,* and *Bc,* then *ABCD* will contain nt such areas as *Abdc.*

If now a particle move with a uniform velocity of n feet a second, and *AC* represent t seconds, on the scale of one inch to

a second, the parallelogram *Bc* will represent the space travelled
over in the first second, since it contains *n* times the parallelo-

gram *Abdc*, and *ABDC* will represent the space travelled over
in *t* seconds.
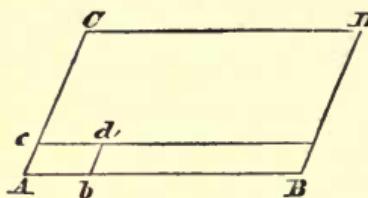
There will be no difficulty in the representation of a period
of time by a line, or of a distance by an area, if the student
bear in mind that periods of time and lengths of lines, although
existing absolutely, are only estimated by their ratios to certain
standard periods, and standard lengths, and they are therefore
determined whenever these ratios are given, either directly in
numbers or by the comparison of any magnitudes whatever *of
the same kind*.

47. Cor. 1, 2. If bodies describe orbits under the action
of certain forces, and small forces, extraneous to those under the
action of which the orbits are described, be supposed to act upon
the bodies, the orbits will be disturbed slightly, and the errors
spoken of are the linear disturbances of the bodies, at any time,
from the positions which they would have occupied at that time,
if the extraneous forces had not acted.

Thus, in calculating the motion of the Moon considered as
moving under the attraction of the Sun and Earth, it is conve-
nient to estimate the motion which she would have, if subjected
to the attraction of the Earth alone, and then to calculate what
would be the disturbing effect of the Sun upon this orbit.

48. If *AB* be a portion of an orbit described by a body in
any time, *AC* the portion of the orbit described when a disturb-
ing force is introduced, *BC* is " quam proxime" the space which
would have been described in the same time from rest by the
action of the disturbing force alone. When the time is taken
small, but not *indefinitely* small, the expression in the statement

of the corollaries, "approximately," is necessary for two reasons; for, in the first place, the position of the body in space is not the same at the end of any interval in the lapse of the time as if the body had moved from rest under the action of the disturbing force alone, and therefore the magnitude of the force is not generally the same either in direction or magnitude; and, in the second place, since the force is not generally uniform, the variation according to the duplicate ratio of the times is not exact, except in the limit.

But, when the times are taken very small, the variation of direction and magnitude of the force may be neglected, as an approximation to the true state of the case.

49. *Application of the method of Lemma X to determine the space described in a finite time from rest by a particle under the action of a constant force.*

Let f be the measure of the acceleration caused by the constant force, so that at the time t the velocity $V = ft$.

Since the velocity varies as the time, the curve Ak in the figure of the Lemma is a straight line, $dD : AD$ being constant.

Therefore the space which is described in the time t, represented by AK, is represented by the area of the triangle AKk or $\frac{1}{2}Kk.AK$. The space described in time t from rest is therefore $\frac{1}{2}Vt = \frac{1}{2}ft^2$.

50. *General geometrical representation of the space described by a body when it moves with a variable velocity for a finite time.*

PROP. If a curve be found, such that the ordinate at each point represents the velocity corresponding to a time represented by the abscissa, then the space described by the body will be represented by the area bounded by the curve, the line of abscissæ, and the ordinates corresponding to the commencement and end of the time of motion.

Let OA, OB represent the times at the commencement and end of the interval during which the motion of the body is to be examined. Let OM be any other time, and let AC, MP, BD, perpendicular to OAB, represent the velocities at the ends of

the times represented by *OA*, *OM*, *OB*; *CPD* the curve which passes through the extremities of all such ordinates as *MP*.
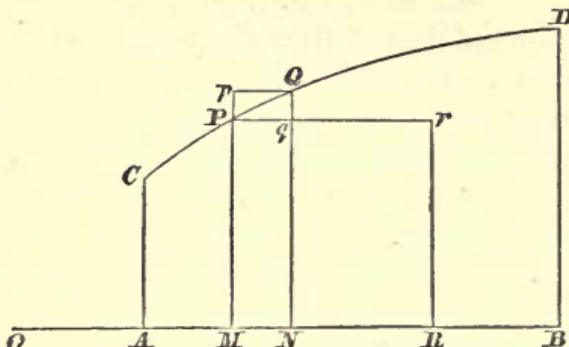
Let *AB* be divided into any number of small portions, such as *MN*; and let *NQ* be the ordinate corresponding to *ON*. Complete the parallelograms *PMNq*, *QNMp*, and suppose corresponding parallelograms to be constructed on all the bases corresponding to *MN*.

The body during the time represented by *MN* moves with a velocity, which, if *MN* be taken small enough, will be intermediate in magnitude between the velocities represented by *PM* and *QN*, and the space described during that time will be intermediate in magnitude between the spaces which would have been described with uniform velocity represented by *PM* and *QN*, or between the spaces represented by the areas *PN*, *QM*.

Hence the whole space described in the interval of time represented by *AB* is greater than that represented by the inscribed series and less than that by the circumscribed series of parallelograms, and each of these is, by Lemma II., ultimately equal to the area *ACDB*, when the number of portions into which *AB* is divided is indefinitely increased, and their magnitudes diminished; therefore the proposition is proved.

51. Cor. 1. Since the area *PMNQ* is ultimately equal to the rectangle *PM.MN*, it follows that *the measure of the velocity at any time is the limit of the quotient of the space described after that time by the time of describing it.*

52. Cor. 2. Let *MR* represent the unit of time, and complete the parallelogram *PMRr*; then the area *PMRr* represents

the space which would be described in an unit of time with a velocity represented by PM; whence it follows that *the velocity of a body at any instant may be measured by the space which it would describe if it moved with that velocity unchanged for an unit of time.*

Measures of Variable Force, Kinetic Energy, Work of a Force.

53. When a particle of mass m is moving in a straight line under the action of an uniform force F, if V, v be the velocities at the beginning and end of the interval of time t, and s be the space described in that time, the following equations will hold: $m(v-V) = Ft$ and $\frac{1}{2}m(v^2 - V^2) = Fs$.

These equations represent respectively that:

(1) The *increase of momentum* in a given *time* is equal to the *whole force* which has acted during that time.

(2) *Half the increase of vis viva,* or the *increase of the kinetic energy* in a given *space* is equal to the *work of the force* in that space.

If F be a variable force, and F_1, F_2 be its least and greatest values during the time t, $m(v-V)$ will be greater than $F_1 t$ and less than $F_2 t$, each of which will become Ft ultimately when t is indefinitely diminished; and similarly for $\frac{1}{2}m(v^2 - V^2)$.

Hence we obtain two measures of variable force in the form of the two limits:

(1) *The quotient of the increase of the momentum by the time, when the time is diminished indefinitely.*

(2) *The quotient of the increase of the kinetic energy by the space through which the force has acted, when that space is diminished indefinitely.*

54. In the velocity curve, Art. 50, the velocity Qq is added in the time MN, the measure of the acceleration at the time OM is therefore the limit of the ratio $Qq : Pq$, or the trigonometrical tangent of the angle which the tangent at P to the velocity curve makes with the line of abscissæ.

55. *Geometrical representation of the momentum generated*

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by a finite and variable force acting for a finite time upon a
particle moving in the direction of the action of the force.

In the figure of p. 72, let OA, OB represent the times at
the commencement and end of the interval during which the
action of the force is considered.

Let AB be divided into any number of small portions, such
as MN, and let PM, QN, perpendiculars to AB, represent the
forces acting on the particle at the times OM, ON respectively,
and let parallelograms be constructed and the curve drawn as
in Art. 50.

The momentum generated in the time MN, if MN be taken
small enough, will be intermediate between the momenta re-
presented by the parallelograms PN and QM; therefore, by
Lemma II., the whole increase of momentum is represented
by the area $ACDB$ bounded by the curve, the line of abscissæ,
and the ordinates at the commencement and end of the finite
interval of time represented by AB.

56. As in Arts. 51, 52, the measure of force given in (1)
Art. 53 can be deduced; also that the force at any instant may
be measured by the momentum which would be generated if
the force were to continue unchanged for an unit of time.

57. *Geometrical representation of the kinetic energy generated*
by a force which acts upon a particle moving in the direction
of the force's action through a finite space.

Let OAB be the line of motion of the particle, and when
it arrives at M let PM perpendicular to OAB represent the
force, and let the construction be made as before.

The increase of kinetic energy in the passage from M to
N is intermediate between the work done by the forces re-
presented by PM and QN, *i.e.* it is represented by an area
which is intermediate between PN and QM; therefore, by
Lemma II, the increase of kinetic energy or the work of the
force during the motion from A to B is represented by the
area $ACDB$.

58. The measure of force given in (2), Art. 53, is deducible
as before, since $PM.MN =$ area $PMNq$ ultimately.

59. *In rectilinear motion of a particle under the action of any variable force, the sum of the kinetic and potential energies is constant.*

If the motion of the particle be considered only within the limits A, B, the area $PMBD$ represents the whole work which the force will be able to do as the particle moves from M to the end of its path; this work is called the *Potential Energy*, and since the kinetic energy at M is represented by the area $CAMP$, it follows that throughout the motion the sum of the kinetic and potential energies is constant.

Application to the determination of the motion of a particle under various circumstances.

(1) *To find the space travelled over in a given time by a body moving with a velocity which varies as the square of the time from the beginning of the motion.*

Let AB represent the time, and let BC perpendicular to AB represent the velocity at the end of that time.
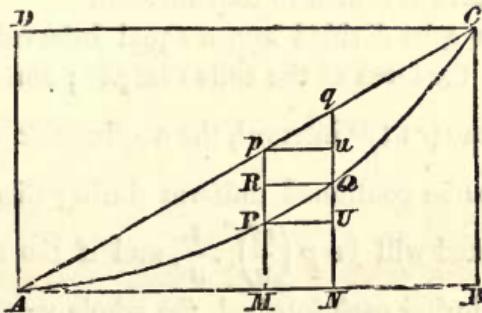
Let AB be divided into any number of equal portions of which MN is one, and let MP, NQ represent the velocities at the ends of the times represented by AM, AN.

Then, since $MP : NQ : BC :: AM^2 : AN^2 : AB^2$, a parabola can be described touching AB and passing through P, Q, C and the extremities of all ordinates by which velocities are represented.

Hence the space described in the time represented by AB is represented by the parabolic area ABC or $\frac{1}{3}AB.BC$.

And if p be the velocity at end of $1''$, pt^2 will be that at

the end of t''; therefore $\frac{1}{3}pt''.t = \frac{1}{3}pt^3$ will be the space described in the time t.

NOTE. The following method of representing the space serves to illustrate Art. 46.

Join AC, and let pM, qN be the ordinates, and suppose the figure to revolve round AB; pM generates a circle whose area $\propto pM^2 \propto AM^2$; therefore this circle may be taken to represent the velocity at the time corresponding to AM, and the solid generated by $pqNM$ represents the space described in time MN. The whole space is therefore represented by the cone generated by ABC, or $\frac{1}{3}AB.\pi BC^2$, which gives the same result as before.

(2) *To find the space described from rest at any time by a particle under the action of a force whose accelerating effect varies as the mth power of the time.*

This problem is more simply solved by applying directly the method of summation, since in order to find the area of the curve, constructed as in Lemma X., we should eventually be obliged to have recourse to that method.

Let the time t be divided into n equal intervals, and let the acceleration by the force at the time t be pt^m; hence, at the commencement of the $(r+1)^{\text{th}}$ interval, the acceleration will be $p\left(\dfrac{rt}{n}\right)^m$, and, if the force be continued uniform during this interval, the velocity generated will be $p\left(\dfrac{rt}{n}\right)^m.\dfrac{t}{n}$, and if the same arrangement be made during each interval, the whole velocity generated will be $\dfrac{1^m + 2^m + \ldots + (n-1)^m}{n^{m+1}}pt^{m+1}$; hence, when the number of intervals is increased indefinitely, it follows, by the reasoning of Lemma II., that the velocity at the time $t = \dfrac{pt^{m+1}}{m+1}$.

In the same manner, if the velocity at the commencement of each interval were continued uniform during the interval, the space described could be shewn to be

$$\frac{1^{m+1} + 2^{m+1} + \ldots + (n-1)^{m+1}}{n^{m+2}} \cdot \frac{pt^{m+2}}{m+1};$$

whence, proceeding to the limit, the space described in the time $t = \dfrac{p t^{m+2}}{(m+1)\,(m+2)}$.

(3) *To find the velocity acquired from rest, when a body is acted on by an attractive force whose accelerating effect varies as the distance from a fixed point.*

Let S be the fixed point, A the point from which the motion commences, and let AB, perpendicular to SA, represent the accelerating effect of the force at A. Join SB, and let MP, per-

pendicular to SA, meet SB in P; then, since $PM : BA :: SM : SA$, PM represents the accelerating effect of the force at M, and the square of the velocity acquired at M is represented, Art. 57, by twice the area $BAPM$ or $SA \cdot AB - SM \cdot MP$.
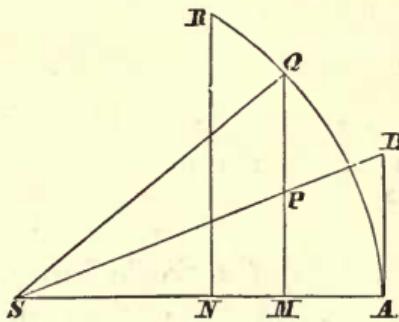
With centre S and radius SA describe a circle AQR, and let MPQ, NR be ordinates at Q, R; then, if μD be the measure of the accelerating effect of the force at a distance D, (vel.)^2 at $M = \mu\,(SA^2 - SM^2)$; therefore the velocity at $M = \sqrt{(\mu)}\, QM$.

(4) *Time of describing a given space from rest under the action of a force varying as the distance from a fixed point.*

The time of describing MN is ultimately, when MN is in-indefinitely diminished, $\dfrac{MN}{\sqrt{(\mu)}\,QM} = \dfrac{QR}{\sqrt{(\mu)}\,SQ} = \dfrac{1}{\sqrt{(\mu)}} \times$ circular measure of QSR; therefore, if t be the time from A to M, $t\sqrt{(\mu)}$ will be the circular measure of ASQ.

Let $SA = a$, then the distance from S at the time $t = a \cos\{t\,\sqrt{(\mu)}\}$, and the velocity $= a\,\sqrt{(\mu)}\,\sin\{t\,\sqrt{(\mu)}\}$; hence, when $t\,\sqrt{(\mu)} = \pi$,

the particle will come to rest at the point A' on the opposite side of S, where $SA' = SA$, and, the time of oscillation from rest to rest, being $\dfrac{\pi}{\sqrt{(\mu)}}$, will be independent of the distance from which the motion commences.

(5) *Simple harmonic motion.*

DEF. The motion of a particle oscillating under the action of a force tending to a fixed point, and varying as the distance from it, is called *simple harmonic motion.*

From the preceding propositions the following construction for simple harmonic motion, which may also be taken as a definition, is obtained.

When a point Q moves uniformly in a circle, and an ordinate QM is drawn from its position at any instant to any diameter AA', the motion of M, the foot of the ordinate, is *simple harmonic motion.**

DEF. The *amplitude* of a simple harmonic motion is the range SA or SA' on each side of the centre.

The *period* is the time which elapses from any instant until the moving point again moves in the same direction through the same position.

(6) *A particle is subject to the action of a force, whose accelerating effect varies as the distance from a fixed point, in the direction of which it acts, the particle is projected from a given point in a direction perpendicular to the direction of the force at that point, to find the path described by the particle.*

Let the force tend to C, and let A be the point of projection, P the position of the particle at any time.

Let CB, perpendicular to CA, be the distance in which a particle would be reduced to rest, if projected from C with the velocity of projection; so that if V be the velocity of projection, and μCP be the accelerating effect of the force at P, $V^2 = \mu CB^2$ by (3).

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* Thomson's and Tait's *Natural Philosophy*, Art. 53.

Describe circles Bb, Aa having the common centre C, and draw CpP' cutting the circles in p and P', and draw pn perpendicular to CB, and pm, $P'M$ to CA.
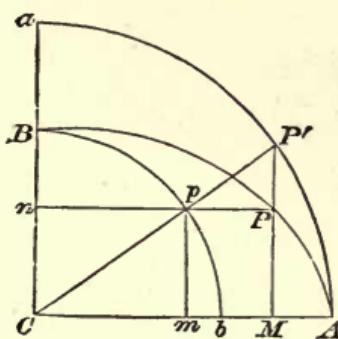
Referring to (4) supra, it will be seen that two particles starting respectively one from rest at A and the other with the velocity of projection at C, under the action of the same force, would arrive simultaneously at M and n, since the time in both cases is proportional to the angle $P'CA$.

But the particle in the proposed problem is acted on at P by a force which is represented by PC, whose accelerating effect parallel to AC and CB is represented by MC and PM, therefore the acceleration in AC is the same as that of the particle supposed to move in AC from rest, and the retardation parallel to BC the same as that of the particle in CB, projected from C, therefore P is in the intersection of np and MP', and $PM : P'M :: pm : P'M :: Cp : CP' :: CB : CA$; therefore the required path of the particle is an ellipse whose semi-axes are CA and CB.

Cor. 1. Area $ACP \varpropto$ area $ACP' \varpropto \angle ACP' \varpropto$ time from A to P, hence the area swept out by the radius vector is proportional to the time.

Cor. 2. The square of the velocity at P is the sum of the squares the velocities of the particles at M and $n = \mu \cdot PM^2 + \mu \cdot pn^2 = \mu \cdot CD^2$, where CD is the semi-diameter conjugate to CP.

(7) *The space described by a body moving in a medium, in which the resistance varies as the velocity, when no other force acts on the body, varies as the velocity destroyed.*

Let the time AK be divided into equal intervals AB, BC, $CD, \ldots$; and let Aa', Bb', ... be the velocities at the beginning of the intervals, the space in time AK is represented by the area $a'AKk'$.
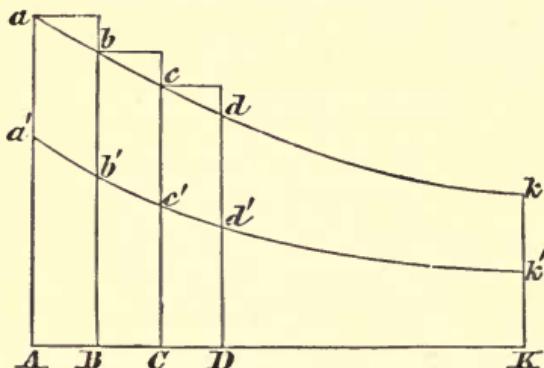
Suppose the force of resistance to be constant throughout the intervals of time AB, BC, ..., and equal to the amount at the commencement of each, and let Aa, Bb, ... be the measures of the retarding effect of those forces, then the velocity destroyed s represented by the limit of the sum of the parallelograms aB, bC, ... or the area $aAKk$; hence the space described and the velocity destroyed vary respectively as the areas $a'AKk'$ and $aAKk$; and, since the resistance varies as the velocity, the ratios $Aa' : Aa$, $Bb' : Bb$, &c., are all equal; therefore, by Lemma IV., the areas $a'AKk'$, $aAKk$ are in a constant ratio; hence the space described varies as the velocity destroyed.

X.

1. If the square of the velocity of a body be proportional to the space described from rest, prove that the accelerating force is constant.

2. At what point of the proof of Lemma X. is it assumed that the body starts from rest?

3. State the proposition by which Lemma X. is replaced, when the body, instead of starting from rest, commences its motion with a given velocity.

4. If a body move from rest under the action of a force which varies as the square of the time from the beginning of the motion, shew that the velocity at any time will vary as the cube of the time, and the space described as the fourth power of the time.

5. If the velocity after a time t from rest be equal to $a(2t + t^2)$, what will be the shape of the curve in the figure, and the space described in any time?

6. If the square of the velocity of a moving point vary as the time, find the space which will be described in a given time; and shew that the acceleration will vary inversely as the velocity.

7. If the curve employed in the proof of the Lemma be an arc of a parabola, the axis of which is perpendicular to the straight line on which the time is measured, prove that the accelerating effect of the force will vary as the distance from the axis of the parabola.

XI.

1. If in the velocity curve of Lemma X. there should occur a point where the two parts of the curve cut one another at a finite angle, what would be the interpretation of this singularity? Explain also what a point of inflexion would imply.

2. A particle is placed in the line joining two centres of attracting force, the accelerating effect of each of which varies as the distance, find the time in which the particle oscillates.

3. When a body moves from rest at A under the action of a force which varies as the square of the distance from $S (= \mu . SM^2$ at M), the square of the velocity at $M = \frac{2}{3}\mu (SA^3 - SM^3)$.

4. If a body be acted on from rest by a repulsive force which varies as the distance from a fixed point, find the velocity when the body arrives at any position.

5. Two points move from rest in such a manner that the ratio of the times in which the same uniform acceleration would generate their respective velocities at those times is constant. Shew that their respective accelerations, at any time bearing that ratio, are equal.

6. Two forces reside at S, one attractive and whose accelerating effect on a particle varies as the distance from S, and the other constant and repulsive; prove that, if a particle be placed at S, it will move until it be brought to rest at a point which is double the distance from S at which it would rest in equilibrium under the action of the forces.

7. A particle moves from rest at A under the action of a force tending to S, and varying as the distance from S, and in its path towards S it strikes another particle of equal mass at rest at B; prove that, if the particles be perfectly elastic, they will meet again on the opposite side of S at a distance equal to SB, and continue to impinge at B and B' for ever.

LEMMA XI.

The vanishing subtenses of the angle of contact, in all curves which have finite curvature at the point of contact, are ultimately in the duplicate ratio of the chords of the conterminous arcs.

Case 1. Let AB be the arc of a curve, AD its tangent at A, BD the subtense of the angle of contact, BAD, perpendicular to the tangent, AB the chord of the arc.

Draw AG, BG perpendicular to the tangent AD and the chord AB respectively, meeting in G; then let
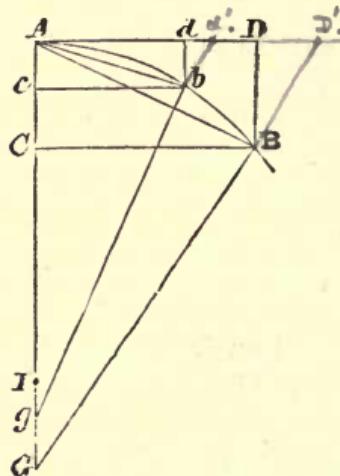
the points D, B, G move towards the points d, b, g, and let I be the point of ultimate intersection of the lines BG, AG, when the points B, D move up to A.

It is evident that the distance GI may be made less than any assigned distance by diminishing AB.

But, since the angles ABD and GAB are equal, and also the right angles BDA, ABG, the triangles ABD, GAB are similar; therefore $BD : AB :: AB : AG$, or $BD.AG = AB^2$, and, similarly, $bd.Ag = Ab^2$;

$$\therefore \ AB^2 : Ab^2 = BD.AG : bd.Ag ;$$

therefore the ratio $AB^2 : Ab^2$ is a ratio compounded of the ratios of $BD : bd$ and $AG : Ag$.

But, since GI may be made less than any assigned length, the ratio $AG : Ag$ may be made to differ from a ratio of equality less than by any assigned difference; therefore the ratio $AB^2 : Ab^2$ may be made to differ from the ratio $BD : bd$ less than by any assigned difference.

Hence, by Lemma I., the ultimate ratio $AB^2 : Ab^2$ is the same as the ultimate ratio of $BD : bd$.   Q.E.D.

Case 2. Let now the subtenses BD', bd' be inclined at any given angle to the tangent; then, by similar triangles $D'BD$, $d'bd'$, $BD' : bd' :: BD : bd$, but ultimately $BD : bd :: AB^2 : Ab^2$; therefore ultimately $BD' : bd' :: AB^2 : Ab^2$.   Q.E.D.

Case 3. And, although the angle D' be not a given angle, if BD' converge to a given point, or be drawn according to any other [fixed] law [by which the angle D' remains finite, since BD' is a subtense], still the angles D', d', constructed by this law common to both, will continually approach to equality and become nearer than by any assigned difference, and will be therefore ultimately equal, by Lemma I., and hence BD', bd' will be ultimately in the same ratio as before.   Q.E.D.

COR. 1. Hence, since the tangents AD, Ad, the arcs AB, Ab and their sines BC, bc become ultimately equal to the chords AB, Ab, their squares also will be ultimately as the subtenses BD, bd.

COR. 2. The squares of the same lines also will be ultimately as the sagittæ of the arcs, which bisect the chords, and converge to a given point; for those sagittæ are as the subtenses BD, bd.

COR. 3. And therefore the sagittæ will be ultimately in the duplicate ratio of the times in which a body describes the arcs with a given velocity.

COR. 4. The rectilinear triangles ADB, Adb are ultimately in the triplicate ratio of the sides AD, Ad,

and in the sesquiplicate ratio of the sides DB, db; since these triangles are in the ratio compounded of $AD : Ad$ and $BD : bd$. So also the triangles ABC, Abc will be ultimately in the triplicate ratio of the sides BC, bc. The sesquiplicate ratio may be regarded as the subduplicate of the triplicate, or as compounded of the simple and the subduplicate ratios.

Cor. 5. And, since DB, db are ultimately parallel and in the duplicate ratio of AD, Ad [therefore, this being a property of a parabola,] at every point at which a curve has finite curvature an arc of a parabola can be drawn which will ultimately coincide with the curve; and the curvilinear areas ADB, Adb will be ultimately two-thirds of the rectilinear triangles ADB, Adb; and the segments AB, Ab the third parts of the same triangles. And hence these areas and these segments will be in the triplicate ratio as well of the tangents AD, Ad as of the chords and arcs AB, Ab.

SCHOLIUM.

But, in all these propositions, we suppose the angle of contact to be neither infinitely greater nor infinitely less than the angles of contact which circles have with their tangents; that is, that the curvature at the point A is neither infinitely great nor infinitely small; in other words, that the distance AI is of finite magnitude.

For DB might be taken proportional to AD^3, in which case no circle could be drawn through the point A between the tangent AD and the curve AB, and the angle of contact would be infinitely less than that of any circle.

And, similarly, if different curves be drawn in which DB varies successively as AD^4, AD^5, AD^6, &c., a series of angles of contact will be presented which may be continued to an infinite number, of which each will

be infinitely less than the preceding. And if curves be drawn in which DB varies as AD^2, $AD^{\frac{3}{2}}$, AD, $AD^{\frac{4}{3}}$, $AD^{\frac{5}{3}}$, &c., another infinite series of angles of contact will be obtained, of which the first will be of the same kind as in the circle, the second infinitely greater, and each infinitely greater than the preceding. But, moreover, between any two of these angles an infinite series of other angles of contact can be inserted, of which each may be infinitely greater or infinitely less than any preceding; for example, if between the limits AD^2 and AD^3 there be inserted $AD^{\frac{6}{5}}$, $AD^{\frac{5}{4}}$, $AD^{\frac{4}{3}}$, $AD^{\frac{7}{5}}$, $AD^{\frac{3}{2}}$, $AD^{\frac{8}{5}}$, $AD^{\frac{5}{3}}$, $AD^{\frac{7}{4}}$, $AD^{\frac{9}{5}}$, &c. And, again, between any two angles of this series there can be inserted a new series of intermediate angles differing from one another by infinite intervals. Nor does the nature of the case admit any limit.

The propositions which have been demonstrated concerning curved lines and the included areas are easily applied to curved surfaces and solid contents.

These Lemmas have been premised for the sake of escaping from the tedious demonstrations by the method of *reductio ad absurdum*, employed by the old geometers. The demonstrations are certainly rendered more concise by the method of indivisibles; but, as there is a harshness in the hypothesis of indivisibles, and on that account it is considered to be an imperfect geometrical method, it has been preferred to make the demonstrations of the following propositions depend on the ultimate sums and ratios of vanishing quantities and on the prime sums and ratios of nascent quantities, *i.e.* on the limits of sums and ratios; and therefore to premise demonstrations of those limits as concise as possible. By these demonstrations the same results are deducible as by the method of indivisibles; and we may employ the principles which have been established with greater safety. Consequently, if, in what follows, quantities

should be treated of as if they consisted of particles [indefinitely small parts], or small curve lines should be employed as straight lines, it would not be intended to convey the idea of indivisible, but of vanishing divisible quantities, not that of sums and ratios of determinate parts, but of the limits of sums and ratios; and it must be remembered that the force of such demonstrations rests on the method exhibited in the preceding Lemmas.

An objection is made, that there can be no ultimate proportion of vanishing quantities; inasmuch as before they have vanished the proportion is not ultimate, and when they have vanished it does not exist. But by the same argument it could be maintained that there could be no ultimate velocity of a body arriving at a certain position at which its motion ceases; for that this velocity, before the body arrives at that position, is not the ultimate velocity; and that, when it arrives there, there is no velocity. And the answer is easy: that, by the ultimate velocity is to be understood that, when the body is moving, neither before it reaches the last position and the motion ceases nor after it has reached it, but at the instant at which it arrives; *i.e.* the very velocity *with which* it arrives at the last position and *with which* the motion ceases.

And, similarly, by the ultimate ratio of vanishing quantities is to be understood the ratio of the quantities, not before they vanish nor after, but *with which* they vanish. Likewise, also, the prime ratio of nascent quantities is the ratio *with which* they begin to exist. And a prime or ultimate sum is that *with which* it begins to be increased or ceases to be diminished.

There is a limit which the velocity can attain at the end of the motion, but cannot surpass. This is the ultimate velocity. And the like can be stated of the limit of all quantities and proportions commencing or ceasing to exist. And, since this limit

is certain and definite, to determine it is strictly a geometrical problem. And all geometrical propositions may be legitimately employed in determining and demonstrating other propositions which are themselves geometrical.

It may also be argued that, if the ultimate ratios of vanishing quantities be given, the ultimate magnitudes will also be given, and thus every quantity will consist of indivisibles, contrary to what Euclid has demonstrated of incommensurable quantities, in his tenth book of the *Elements*.

But this objection rests on a false hypothesis. Those ultimate ratios with which quantities vanish are not actually ratios of ultimate quantities, but limits to which the ratios of quantities decreasing without limit are continually approaching; and which they can approach nearer than by *any* given difference, but which they can never surpass, nor reach before the quantities are indefinitely diminished

The argument will be understood more clearly in the case of infinitely great quantities. If two quantities, of which the difference is given, be increased infinitely, their ultimate ratio will be given, namely, a ratio of equality, yet in this case the ultimate or greatest quantities of which that is the ratio will not be given.

In what follows, therefore, if at any time, for the sake of facility of conception, the expressions *indefinitely small*, or *vanishing*, or *ultimate* be used concerning quantities, care must be taken not to understand thereby quantities determinate in magnitude, but to conceive them in all cases quantities to be diminished without limit.

Curvature of Curves.

60. The curvature of a curve at any point is greater or less as the amount of deflection from the tangent at that point, in the immediate neighbourhood of the point, is greater or less.

Two curves will have the same curvature at two points, taken one in each, if at equal distances from the points of contact, in the immediate neighbourhood of the points, they have the same deflection from the tangents at those points.

61. An exact geometrical test of equality of curvature may be obtained as follows:

If AB, ab be two curves which have the same curvature at A, a respectively, draw the tangents AC, ac and take $AC = ac$.
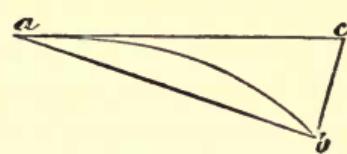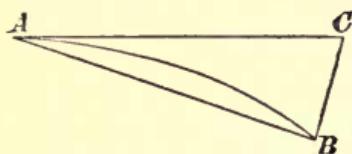
Draw subtenses BC, bc inclined at equal angles to the tangents.

If BC and bc were equal, for all equal values of AC, ac, the curves would be equal and similar. If $BC : bc$ be ultimately a ratio of equality, when AC, ac are taken indefinitely small, the curves will have the same deflection from the tangents in the immediate neighbourhood of A, a, or the curves will have the same curvature at those points.

If the chords AB, ab be drawn, it will be an immediate consequence that the ultimate ratio of the angles BAC, bac will be a ratio of equality. These angles are called *the angles of contact*.

Hence, curves will have the same curvature at two points, one in each, if, equal tangents being drawn at those points, and subtenses inclined at any equal angles to the tangents, the limiting ratio of the subtenses be a ratio of equality, or if the limiting ratio of the angles of contact be a ratio of equality.

62. The curvature of one curve will be infinitely greater or infinitely less than that of another if the limiting ratio of the subtense of the first to that of the second be infinitely great or infinitely small.

63. The ratio of the curvature of one curve to that of another at two points, or of the curvature of the same curve at two different points, is the limiting ratio of the subtenses drawn from the extremities of equal tangents and inclined at equal angles to the tangents.

64. The curvature of a curve is said to be finite, at any point, when the ratio of the curvature at that point to that of any circle whose radius is finite, is a finite ratio.

65. *The curvature of a circle is the same at every point.*

Let A, a be any two points on a circle, AC, ac equal tangents at A, a, Cb, cb subtenses perpendicular to the tangents, OD, Od perpendicular to the subtenses produced; therefore $CD = cd$, each being equal to the radius, and $BD = bd$; hence $BC = bc$ always, and therefore ultimately, when the arcs are indefinitely diminished, $BC : bc$ is a ratio of equality; therefore
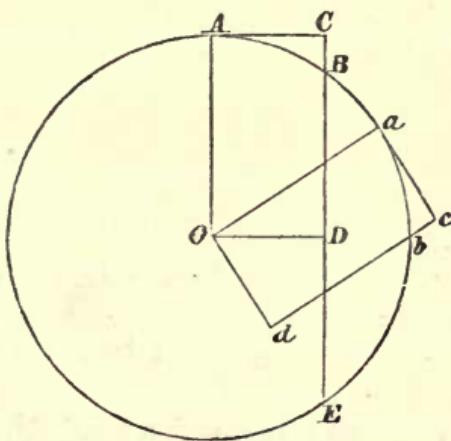
the circle has the same curvature at any two points.

66. *In different circles the curvatures vary inversely as the radii.*

In the last figure, produce CB to the circumference in E. Then, $AC^2 = CB.CE$; also, if $A'C'$ be a tangent to another circle, and $A'C'$ be taken equal to AC, and the same construction be made, $A'C'^2 = C'B'.C'E'$; therefore $CB.CE = C'B'.C'E'$, and $CB : C'B' :: C'E' : CE$; and when AC, $A'C'$ are indefinitely diminished, $CE = 2AO$; therefore $CB : C'B' :: A'O' : AO$, ultimately, or the curvatures are inversely proportional to the radii.

Measure of Curvature.

67. The curvature of a circle is the same at every point; the curvatures of different circles vary inversely as the diameters

of the circles; and a circle can be constructed of any degree of finite curvature by varying the magnitude of the diameter.

Hence, a circle can always be found whose curvature at any point is equal to that of a curve at a fixed point.

The curvature of a curve at any point is therefore completely determined when the diameter of the circle is found, which has the same curvature as the curve at the given point.

The diameter of the circle, which has the same curvature as the curve at a given point, is called *the diameter of curvature of the curve at that point.*

The chord of the circle, drawn in any direction, is called *chord of curvature in that direction.*

The circle itself is called *the circle of curvature,* and is the circle which has the same tangent as the curve at any point, and also the same curvature.

68. Any other curve might have been chosen to establish a standard measure of finite curvature; but, since no curve but the circle has the same curvature at every point, it would then have been necessary, after selecting the curve, to specify the point, the curvature at which might be made the measure of curvature.

Thus, if the standard curve were a parabola, we must choose the curvature of the parabola at the vertex or at the extremity of the latus rectum or at some determinate point, by which to obtain the measure.

The inconvenience is obvious.

General Properties of the Circle of Curvature.

69. If a circle be drawn touching a curve at a given point, and cutting it at a second point, as the second point approaches indefinitely near the point of contact, the circle will assume a limiting magnitude, and will evidently satisfy the condition of having the same curvature as the curve at that point.

70. Since a tangent at any point is the limiting position of a side, terminated in that point, of a polygon inscribed in the curve, when the number of sides is increased indefinitely,

so the circle of curvature at any point is the limiting circle which passes through three consecutive angular points of the polygon, one of which coincides with the point.

71. *No circle can be drawn whose circumference lies between a curve and its circle of curvature, in the neighbourhood of the point at which the circle of curvature is drawn.*

For, let AQ be the arc of the curve, Aq of the circle of curvature; and let, if possible, another circle be drawn, of which the arc AS lies between the curve and circle, and having therefore the same tangent AR at A; and let RQ, the subtense perpendicular to the tangent, cut the circles in S, q.
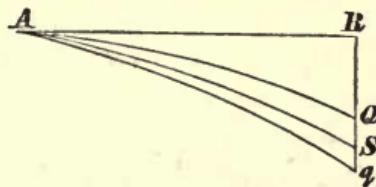
Then $SR : qR$ will be ultimately in the inverse ratio of the diameters of the circles; therefore SR will be ultimately unequal to qR; but, since qR and QR are ultimately in a ratio of equality, SR, which is intermediate in magnitude, will be ultimately equal to either, which is absurd; therefore no circle, &c.

This proposition corresponds to *Euclid* III., Prop. XVI.

72. *The circle of curvature generally cuts the curve.*

For the curvature of the curve at different points taken along the curve continually increases or continually diminishes, until it arrives at a maximum or minimum value.

If therefore the circle of curvature be drawn at any point, on the side on which the curvature is increasing, as we proceed from the point, the curve will lie within the circle, and on the other side, on which the curvature is diminishing, the curve will lie without the circle; which proves the proposition for the general position of the point.

For the particular case, in which the point is at a position of maximum or minimum curvature, as at the extremities of the axes of an ellipse, if the curvature be a maximum the curvature at adjacent points on either side will be less than that of the

circle of curvature at the point under consideration; therefore the circle will lie entirely within the curve on both sides near the point of maximum curvature; and, similarly, it will lie without the curve at points of minimum curvature.

We can illustrate this by reference to the polygon inscribed in the curve; see the figure in the following page.

If, in the curve, equal chords AB, BC, CD, DE,... be placed in order, generally the angles ABC, BCD, CDE,... will increase or decrease, commencing from any point, which property of the polygon will have in the curvilinear limit, when the chords are diminished indefinitely, the corresponding property, that the curvature decreases or increases continually.

Suppose the angles are increasing from B; make the angles CBA', CDE' equal to the angle BCD, and BA', DE' equal to BC, CD...; then a circle through B, C, D will pass also through A' and E', and these points will be on opposite sides of the perimeter of the polygon, whence, if we proceed to the limit, the circle of curvature at a point in the middle of increasing curvature will cut the curve.

If the angles ABC and DEF be each less than the angles BCD, CDE, supposed equal, the curvature will decrease and then increase, and the circle about BCD will pass through E, and BA, EF will lie within the circle, and, proceeding to the limit, the circle of curvature will lie without the curve, near the point of minimum curvature.

Evolute of a Curve.

73. DEF. If the circles of curvature be drawn at every point of a curve, the centres of those circles will lie in a curve which is called the *evolute* of the proposed curve.

Properties of the Evolute.

74. *The extremity of a string unwrapped from the evolute of a curve traces out the curve.*

Let $ABCDE$ be any equilateral polygon, and let $a'a$, $b'b$, $c'c$ $d'd$ be drawn perpendicular to the sides from the middle points

a', b', &c., these intersect in the angular points $abcd$... of another polygon.

If a string were wrapped round $a'abcd$... the extremity a' would as the string was unwrapped pass through the points $a'b'c'd'$.

Let now the number of sides of the polygon be increased and the magnitude diminished indefinitely.

The points $a'b'c'$... will be ultimately in the curve which is the limit of the polygon, and since a, b, c... are the centres of the circles described about ABC, BCD,..., a, b, c,... will be ultimately the centres of the circles of curvature at $a'b'c'$..., and
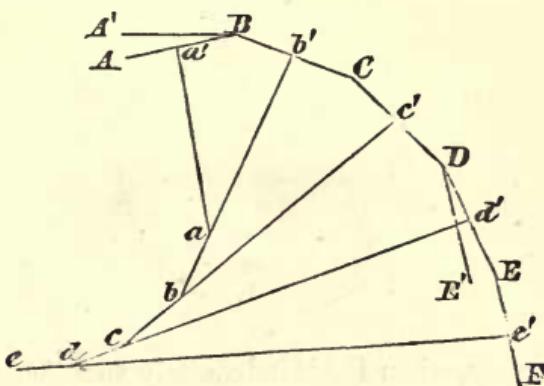
the curve, which is the limit of the polygon $abcd$..., will be the evolute of the curve $a'b'c'$..., and the property proved for the polygons will be true for the limits of the polygons, therefore the extremity of the string unwrapped from the evolute will trace the curve of which it is the evolute. This property gives rise to the name of evolute.

DEF. The curves formed by the unwrapping of a string from a curve are called *involutes*.

75. *The tangent to the evolute of a curve is a normal to the curve.*

Since $b'b$ is ultimately the tangent to the evolute and is perpendicular to BC, which is ultimately the tangent to the curve $a'b'c'$..., therefore the tangent to the evolute is a normal to the curve.

Propositions on Diameters and Chords of Curvature.

76. *If a subtense be drawn from the extremity of an arc of finite curvature, in any direction, the chord of curvature parallel to that direction will be the limit of the third proportional to the subtense and the arc.*

Let PQ, Pq be arcs of a curve and its circle of curvature at P, let PR be the common tangent, and RQq the direction of a common subtense, meeting the circle in U.

Draw the chord PV parallel to RQ. Then, since $Rq.RU=PR^2$, RU is the third proportional to Rq and PR.
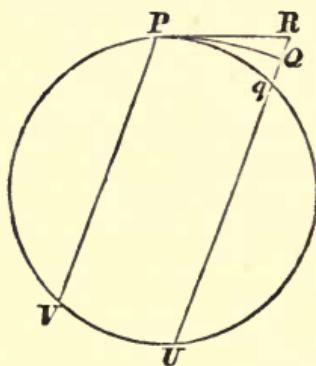
But, ultimately, when PQ is indefinitely diminished, $RU=PV$, and $PR=PQ$, by Lemma VII. also, $Rq=RQ$ by the property of the circle of curvature.

Therefore PV is the limit of the third proportional to RQ and PQ.

Cor. *The diameter of curvature is the limit of the third proportional to the subtense perpendicular to the tangent and the arc.*

77. *The two chords of curvature at any point of a parabola drawn through the focus, and in the direction of the diameter, are each equal to four times the focal distance of that point.*

Let AP be a parabola, P any point, RQ a subtense parallel to the diameter PMx, QM the ordinate at Q, S the focus. Then, by a property of the parabola, $QM^2=4SP.PM$; therefore $4SP$ is a third proportional to PM and QM, *i.e.* to RQ and PR.

Hence, $4SP$ is the limit of the third proportional to the

subtense QR and the arc PQ, and is therefore equal to the chord of curvature at P in direction of the diameter.
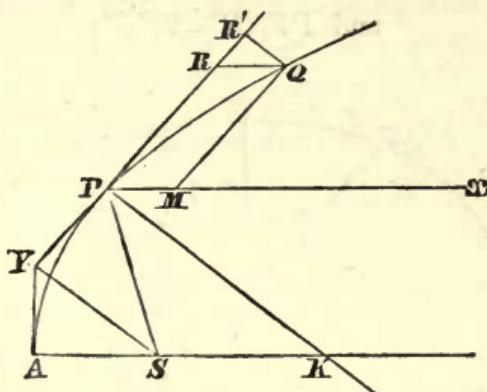
And, since PS, PM are equally inclined to the tangent at P, the chords in those directions are equal; therefore the chord of curvature through S is four times the focal distance SP.

78. *One-fourth of the diameter of curvature at any point of a parabola is a third proportional to the perpendicular from the focus on the tangent at that point, and the focal distance of that point.*

For, draw SY, QR' perpendicular to PR, and let PI be the diameter of curvature at P.

Then $PI.QR' = PQ^2 = PR^2$ ultimately, $= 4SP.QR$;

$$\therefore PI : 4SP :: QR : QR' :: SP : SY;$$

since the triangles SYP, $QR'R$ are similar; therefore $\frac{1}{4}PI$ is a third proportional to SY and SP.

79. *The chord of curvature at any point of an ellipse drawn through the centre of the ellipse is a third proportional to the diameter through that point and the diameter conjugate to it.*

Let P be any point in an ellipse, PCG the diameter, DCD' conjugate to it, Q any point near P, QR a subtense parallel to CP, QM an ordinate parallel to DC, PV the chord of curvature drawn through C.

Then $PV.QR = PQ^2 = QM^2$ ultimately,

and $QM^2 : PM.MG :: CD^2 : CP^2$;

$$\therefore PV.QR : QR.MG :: CD^2 : CP^2 \text{ ultimately};$$

$$\therefore PV : 2CP :: CD^2 : CP^2 \text{ ultimately;}$$
$$\therefore PV.CP : CP^2 :: 2CD^2 : CP^2,$$
$$\text{and } PV.CP = 2CD^2;$$
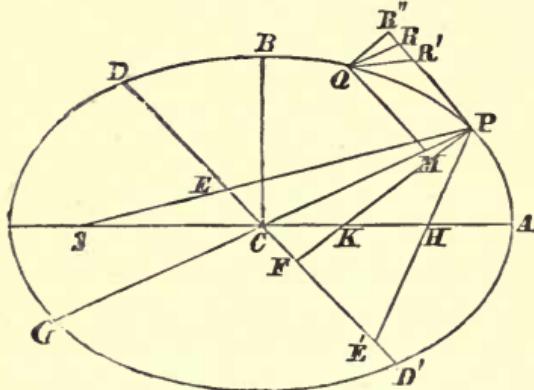
or PV is a third proportional to PG and DCD'.

80. *The chord of curvature at any point through tne focus is a third proportional to the major axis, and the diameter parallel to the tangent at that point.*

Draw the focal distance SP cutting the diameter DCD' in E, let PV' be the chord of curvature through S, and draw the subtense QR' parallel to SP.

$$\text{Then } PV' : PV :: QR : QR' \text{ ultimately}$$
$$:: CP : PE \text{ by similar triangles;}$$
$$\therefore PV'.PE = PV.CP = 2CD^2;$$

$\therefore PV'$ is a third proportional to $2PE$ and DCD', and $2PE$ is equal to the major axis.

Similarly for the other focus H.

81. *The diameter of curvature at any point is a third proportional to twice the perpendicular from the point on the diameter parallel to the tangent and that diameter.*

Draw QR'' perpendicular to the tangent, and PF perpendicular to DCD', and let PI be the diameter of curvature.

$$PI : PV :: QR : QR'' :: CP : PF;$$
$$\therefore PI.PF = PV.CP = 2CD^2;$$

$\therefore PI$ is a third proportional to $2PF$ and DCD'.

82. Since the chord of curvature in any direction varies inversely as the subtense QR drawn in that direction, it is easily seen that, if PL be the portion of the chord intercepted between P and DCD', the chord of curvature at P in the direction PL will be the third proportional to $2PL$ and DCD'.

83. The propositions concerning the chords and diameter of curvature of an ellipse may be proved in the same words for the hyperbola, employing the following figure.
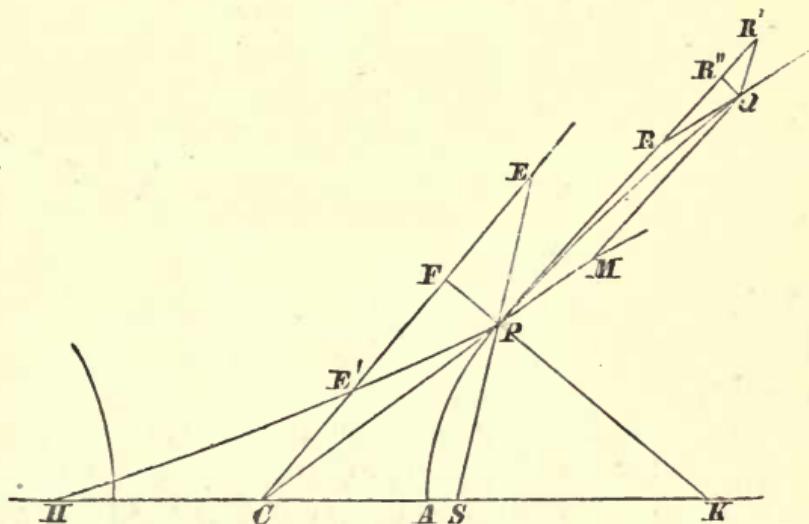
84. *The radius of curvature at any point of a conic section is to the normal in the duplicate ratio of the normal to the semi-latus rectum.*

Let PK be the normal, PO the radius of curvature at P, L the semi-latus rectum.

(i) For the parabola,
$$PO : 2SP :: SP : SY :: SY : SA;$$
$$\therefore PO : 2SY :: SP : SA :: 4SP.SA : L^2;$$
but $PK = 2SY$; $\therefore PK^2 = 4SP.SA$; $\therefore PO : PK :: PK^2 : L^2$.

(ii) For the ellipse or hyperbola,
$$PO.PF = CD^2 \text{ and } PK.PF = BC^2;$$
$$\therefore PO : PK :: CD^2 : BC^2 :: AC^2 : PF^2;$$
but $PF.PK = BC^2 = L.AC$; $\therefore AC : PF :: PK : L$;
$$\therefore PO : PK :: PK^2 : L^2.$$

O

85. *To find the chord common to a conic section and the circle of curvature at any point.*

If a circle intersect a conic section in four points, as $PQUR$, and these points be joined in pairs by two lines, these lines will
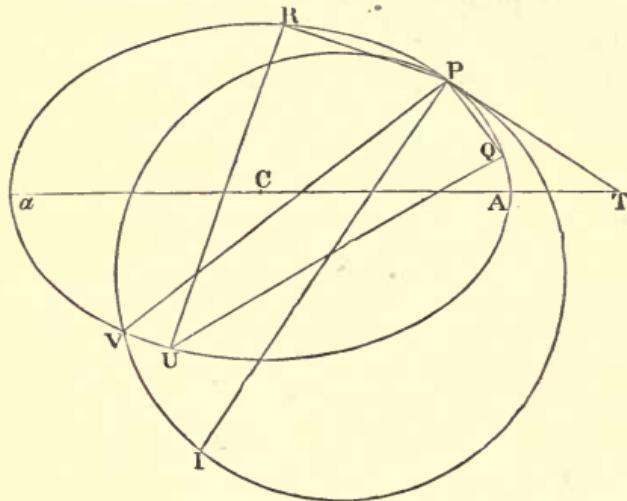
be equally inclined to the axis of the conic section. Thus, in the conic section, PQ, RU are equally inclined to the axis.

For, if UR, QP intersect in O, $OR.OU = OP.OQ$, hence the diameters of the ellipse parallel to UR, QP will be equal, and therefore equally inclined to the axis.

Let Q and R move up to and ultimately coincide with P, then the intersecting circle becomes the circle of curvature at P, and PQ is in the direction PT of the tangent, ultimately, and RU assumes the position of the chord common to the conic section and the circle of curvature at P. Hence, if PV be drawn at an equal inclination with PT to the axis, PV will be the common chord required.

And, if VI be drawn perpendicular to PV, meeting the normal at P in I, PI will be the diameter of curvature at P.

86. *To find the radius of curvature of a curve defined by the relation between the radius vector and the perpendicular from the pole on the tangent.*

Let PY, $PP'Y'$ be the directions of consecutive sides of a polygon inscribed in a curve, SY, SY' perpendiculars on these

sides; draw PO, $P'O$ perpendicular to the same sides, inter-

secting in O, and $P'U$ perpendicular to SP, and let SY, PY' intersect in W.
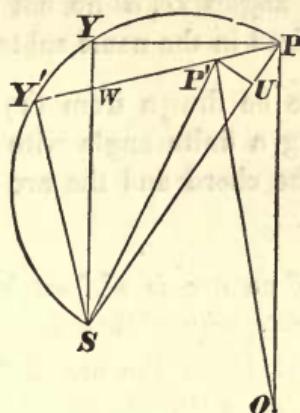
A semicircle on SP as diameter passes through Y and Y';

$\therefore \angle YPW = \angle YSY' = \angle POP'$, and $\angle WYP = \angle OP'P$;

therefore the triangles POP', WPY are similar;

$$\therefore PO : PP' :: PW : YW,$$

$$\text{also } PP' : SP :: PU : PY',$$

by similar triangles $P'UP$, $SY'P$, and $PW = PY'$ ultimately;

$\therefore PO : SP :: PU : YW :: SP \sim SP' : SY \sim SY'$ ultimately.

Also, if PV be the chord of curvature through S,

$$PV : 2PO :: SY : SP;$$

$$\therefore PV : 2SY :: SP \sim SP' : SY \sim SY' \text{ ultimately.}$$

Observations on the Lemma.

87. In the proof of Lemma XI., AI is the limit of the third proportional to BD and AB, hence it is the diameter of curvature of the curve at A.

88. For an example of a law according to which, in Case 3, the directions of the subtenses may be determined, we may suppose that they always pass through a point given in position at a finite distance from A, or that they always touch a given curve; but it must be observed that the case in which they

touch a curve which has the same tangent AD at A is excluded, since in this case the angles D', d' do not in the limit remain finite, a property required in the name subtense.

89. DEF. If a line be drawn from the middle point of an arc of a curve, making a finite angle with the chord, the part intercepted between the chord and the arc is called the *sagitta* of the arc.

90. *The sagitta of an arc is ultimately one quarter of the subtense drawn at the extremity of the arc parallel to the sagitta.*

Let the sagitta FE bisect the arc AB in E, and be produced to the tangent at A in G, and let BD be a subtense parallel to FE.
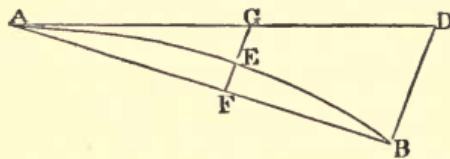
Then $EG : BD :: AE^2 : AB^2$ ultimately ; $\therefore$ $BD = 4EG$,

also $BD : FG :: AD : AG :: AB : AE$ ultimately ;

$\therefore$ $BD = 2FG = 4EG$; hence $FE = EG = \frac{1}{4}BD$ ultimately.

91. COR. 5. The parabola mentioned in this corollary is a parabola of curvature at that point; for, since DB is taken in any given direction, the proportion $BD : bd :: AD^2 : Ad^2$ proves that the curve is ultimately in the form of a parabola, and that, therefore, the line through A drawn in the given direction is the corresponding diameter of the parabola of curvature.

Hence the axis of the parabola may be taken in any assigned direction.

If the subtenses be perpendicular to the tangent, the parabola of curvature will be the parabola whose curvature at the vertex will determine the curvature of the curve, since the axis will be perpendicular to the tangent, and if $4AU$, in the figure page 104, be the third proportional to the subtense and arc, the limiting position of U will be the focus of the parabola.

By means of this corollary the proposition alluded to under Lemma IX., Art. 44, is established; viz. that the ratio of the

areas which takes place of the duplicate ratio, obtained in that Lemma, is the triplicate ratio of the same lines, when the line, AE, instead of cutting the tangent at a finite angle, coincides with the tangent.

92. Scholium. Let AB, AC be two curves, having a common tangent AD at A, and let subtenses DB, DBC of the
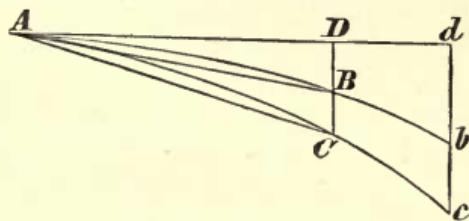
angles of contact be drawn from D at any point in the tangent in the same direction, and let $BD \propto AD^m$, $CD \propto AD^n$ in the curves AB, AC respectively. Draw dbc a common ordinate from a fixed point d, parallel to DBC. Then

$$AD^m : Ad^m :: BD : bd,$$

$$\text{and } AD^n : Ad^n :: CD : cd,$$

and if m be greater than n, $= n + r$ suppose,

$$AD^n . AD^r : Ad^n . Ad^r :: BD : bd;$$

$$\therefore CD . AD^r : cd . Ad^r :: BD : bd$$

$$:: BD . AD^r : bd . AD^r;$$

$$\therefore CD : BD :: cd . Ad^r :: bd . AD^r,$$

and since b, c, d are fixed, and AD vanishes in the limit, therefore CD is indefinitely greater than BD; also, since the angles of contact BAD, CAD are ultimately proportional to BD, CD, it follows that, if in two curves the subtenses vary according to different powers of the arcs or tangents, the angle of contact of that curve in which the index of the power is the least will be infinitely greater than the angle of contact of the other.

Illustrations.

(1) *Two tangents AT, BT are drawn at the extremities of an arc AB, to prove that AT is ultimately equal to BT, when AB is indefinitely diminished.*

Draw $TCUV$ in any direction making a finite angle with the tangents, and meeting the circles of curvature at A and B in UV.
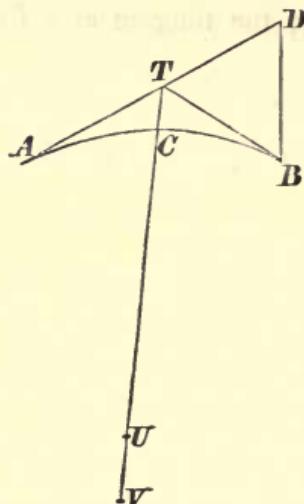
Then since the circle of curvature at A is the limit of the circle which passes through C and has the tangent AT at A, and similarly for that at B, we have ultimately

$$TA^2 : TB^2 :: TC.TU : TC.TV,$$

and $TU = TV$ ultimately; $\therefore TA = TB$ ultimately.

Cor. If BD be any subtense of the arc AB,

$$AT + TB = AB = AD \text{ ultimately};$$

therefore AD will be ultimately bisected by the tangent BT.

(2) *If BT be a tangent at B, AB, BC equal chords of a curve of finite curvature, drawn from B, and AB be produced to c, making $Bc = AB$, and Cc be joined meeting BT in T, cT will ultimately be equal to CT, when the arcs AB, CB are diminished indefinitely.*

Let AU be drawn parallel to CT, meeting the tangent at B in U, and let two circles touch UBT at B and pass one through
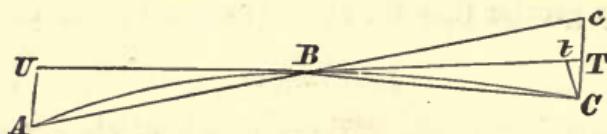
A and the other through C, and let BV, BV' be chords of these circles drawn parallel to AU or CT, then $AU.BV = AB^2$, and

$CT.BV' = BC^2$; but $BV = BV'$ ultimately, since the two circles are each ultimately the circle of curvature at B and $AB = BC$, therefore $AU = CT$ ultimately.

Through B draw RBR' parallel to AC, meeting AU in R' and Cc in R, then $R'U = RT$, therefore $2RT$ is the difference between AU and CT, hence RT ultimately vanishes compared with CT, and since $CR = Rc$, therefore $CT = Tc$ ultimately.

(3) *If, from the point of contact of a curve with its tangent, equal distances be measured along the curve and tangent, the line joining their extremities will ultimately be parallel to the normal at the point of contact.*

In the last figure, let, BC, BT be equal distances, measured along the arc and the tangent; join CT, let the tangent at C meet BT in D, produce BT to F making $DF = DC$, take $BE =$ the chord BC, and join EC, TC, and FC.

Since the arc BC is intermediate in magnitude between $BD + DC$ and BC, therefore, BT being equal to arc BC, the point T lies always between E and F. But the triangles BCE, DCF being both isosceles, each of the angles BEC, BFC will ultimately be a right angle, therefore the angle BTC, which is less than BEC and greater than BFC, will also ultimately be a right angle.

Hence CT will ultimately be parallel to the normal at B.

NOTE. In order to shew the danger of falling into an error by a careless employment of the propositions proved in the first section, the following fallacious proof may be noticed of the above proposition.

In the figure page 102, join BC, then $BT : CB$ will be ultimately a ratio of equality, by Lemma VII; therefore CBT being an isosceles triangle ultimately, CT will be perpendicular to the line bisecting the angle CBT, and therefore to the tangent BT, since BT and BC will ultimately coincide with the bisecting line.

The fact is that Lemma VII. only allows us to assert that BT and the chord BC differ by a quantity Tt, which vanishes compared with either of them, and therefore Tt may $\propto BC^2$; but, by Lemma XI, $CT \propto BC^2$; hence $Tt : CT$ may possibly

be a finite ratio, or CT may be ultimately inclined at any finite angle to BT, at least as far as the reasoning given in the above proof is concerned.

(4) *To construct for the focus of the parabola of curvature whose axis is in a given direction.*
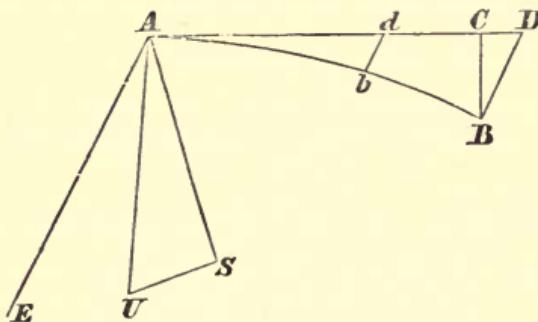
Let AB be a curve of finite curvature, BD, bd subtenses parallel to AE the given direction. Draw AU perpendicular to AD, and AS making angle $UAS = UAE$; then since AE is a diameter of the parabola by Art. 91, AS is in the direction of the focus.

Also, if $4AS$ be taken a third proportional to BD and AD, the limiting position of S will be the focus of the parabola.

(5) *To find the locus of the focus of the parabola of curvature, when its axis changes its direction.*

Let BC be perpendicular to AD, and AU be chosen so that $4AU.BC = AC^2$, then the limiting position of U is the focus of the parabola whose curvature at the vertex is the same as that of the curve at A; also, if S be the focus of the parabola whose axis is parallel to DB, $4AS.DB = AD^2 = AC^2$, ultimately; therefore $AU : AS :: BD : BC$, and $\angle SAU = \angle DBC$; hence if we join SU, the triangles SAU, CBD will be similar, and $\angle ASU = \angle BCD =$ a right angle; therefore the locus of S is a circle on AU as diameter.

(6) ABC *is an arc of finite curvature, and is divided so that* $AB : BC :: m : n$, *a constant ratio; join* AB, AC, BC, *and shew that, ultimately,* $\triangle ABC :$ *segment* $ABC :: 3mn : (m + n)^2$.

For, by Cor. 5, Lemma XI.

$$\text{seg } AB : \text{seg } ABC :: AB^3 : ABC^3 :: m^3 : (m + n)^3$$
$$\text{seg } BC : \text{seg } ABC :: n^3 : (m + n)^3 ;$$
$$\therefore \text{seg } AB + \text{seg } BC : \text{seg } ABC :: m^3 + n^3 : (m + n)^3,$$
$$\text{and } \Delta\, ABC = \text{seg } ABC - \text{seg } AB - \text{seg } BC ;$$
$$\therefore \ \Delta\, ABC : \text{seg } ABC :: 3\,(m^2 n + m n^2) : (m + n)^3$$
$$:: 3mn : (m + n)^2.$$

(7) *To find the chord of curvature, at any point of the cardioid, through the focus.*

It is easily seen from p. 56 (3), that SY being perpendicular to PT, the triangles PSY, pBm, and CBp are similar;
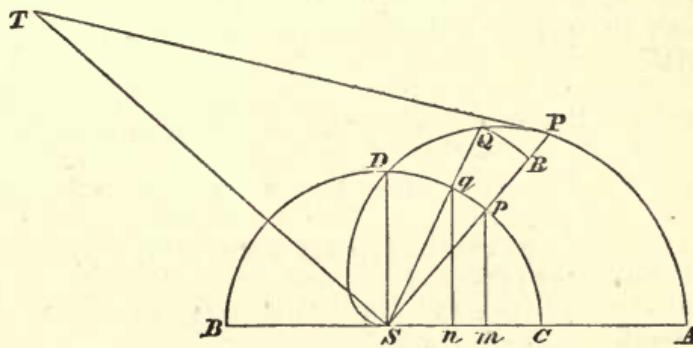
$$\therefore SY : SP :: Bm : Bp :: Bp : BC;$$
$$\therefore SY^2 : SP^2 :: SP : BC, \text{ since } Bm = SP,$$
$$\therefore SY^2 BC = SP^3, \text{ and } (SY^2 - SY'^2)\, BC = SP^3 - SP'^3;$$
$$\therefore SP \sim SP' : SY \sim SY' :: 2SY.BC : 3SP^2 \text{ ultimately};$$
$$\therefore \text{ by Art. 86, chord of curvature} : 2SY :: 2SP : 3SY;$$
therefore the chord of curvature through $S = \tfrac{4}{3} SP.$

XII.

1. Prove that the focal distance of the point in the parabola at which the curvature is one-eighth of that at the vertex is equal to the latus rectum.

2. Prove that the diameter of curvature at the vertex of the major axis of an ellipse is equal to the latus rectum : and shew that the ratio of the curvature at the extremities of the axes is that of the cubes of the axes.

3. Shew that at no point of an ellipse will the circle of curvature pass through the centre, if the eccentricity be less than $\sqrt{\frac{1}{2}}$.

4. Find for what point of an ellipse the circle of curvature passes through the other extremity of the diameter at that point, shew that the distance of this point from the centre is the side of the square of which AB is the diagonal.

5. In a rectangular hyperbola, the diameter of curvature at any point, and the chords of curvature through the focus and centre are in geometrical progression.

6. Prove that at a point P in an ellipse for which the minor axis is a mean proportional between the radius of curvature and the normal, $PC = AC - BC$. Shew that this is impossible unless $AC \gtrless 2BC$.

7. If the radius of curvature for an ellipse at P be twice the normal, prove that $CP = CS$.
If moreover $AC = 2BC$, prove that $CP = 3PM$.

8. If the circle of curvature at a point P of a parabola pass through the other extremity of the focal chord through P, and the tangent at P meet the axis in T, prove that the triangle PST will be equilateral.

9. Prove that the distance of the centre of curvature, at any point of a parabola, from the directrix is three times that of the point.

10. If the circle of curvature at a point on a parabola touch the directrix, the focal distance of the point will be $\frac{9}{16}$ of the latus rectum.

11. PQ is a normal at a point P of a rectangular hyperbola, meeting the curve again in Q, prove that PQ is equal to the diameter of curvature at P.

12. Prove that the portion of the normal intercepted between the line joining the extremities of the two chords of curvature through the foci of an ellipse, and the point of contact P, is $\dfrac{2BC^2}{PF}$.

13. A fixed hyperbola is touched by a concentric ellipse. If the curvatures at the point of contact be equal, the area of the ellipse will be constant.

14. Shew that the directrices of all parabolas touching a curve of finite curvature at any given point, and having the same curvature at that point as the curve, pass through a fixed point.

XIII.

1. Prove that the chord of curvature through the vertex A of a parabola : $2PY :: 2PY : AP$, Y being the intersection of the tangents at P and A.

2. Apply the property that the radius of curvature at any point of an ellipse is to the normal in the duplicate ratio of the normal to the semi-latus rectum, to shew that the radius of curvature at the extremity of the major axis is equal to the semi-latus-rectum.

3. Assuming only that a curve has a subnormal of constant length, prove geometrically that its radius of curvature varies as the cube of its normal.

4. If Pp be any chord of an ellipse, PT, pT tangents at P and p, shew that the curvatures at P and p are as the cubes of pT and PT.

5. Shew that the sum of the chords of curvature through a focus of an ellipse at the extremities of conjugate diameters is constant. Also, if ρ, σ be the radii of curvature at those points, prove that $\rho^{\frac{2}{3}} + \sigma^{\frac{2}{3}}$ is constant.

6. Prove that the chords of curvature through any two points on an ellipse in the direction of the line joining them are in the same ratio as the squares on the diameters parallel to the tangents at the points.

7. Prove that the distances of the centre of curvature at any point of an ellipse and of that point from the minor axis are in the duplicate ratio of the distances of the point and the directrix from the same axis.

8. An hyperbola touches an ellipse, having a pair of conjugate diameters of the ellipse for its asymptotes. Prove that the curves have the same curvature at the point of contact.

9. Shew that, if D be the diameter of an ellipse parallel to the tangent at a point P, whose eccentric angle is ϕ, the length of the chord common to the ellipse and circle of curvature at P will be $D \sin 2\phi$

10. Determine a parabola of curvature in magnitude and position for any point in a circle, when the subtenses are inclined at 45° to the tangent.

11. If x, y be the coordinates of a point P of a curve OP, passing through the origin O, the diameter of curvature at O will be $\dfrac{x^2 + y^2}{x \sin a \sim y \cos a}$ ultimately, a being the inclination of the tangent at O to the line of abscissæ. Hence shew that, if the equation of a curve, referred to rectangular areas, be $y^2 + 2ay - 2ax = 0$, the radius of curvature at the origin will be $2\sqrt{2}.a$.

12. A circle is a circle of curvature, at a fixed point in the circumference, to an ellipse, one focus of which lies on the circle, shew that the locus of the other focus is also a circle.

13. Prove that the chord of curvature at any point P of an ellipse in any direction PQ is half the harmonic mean between the two tangents drawn from P to the confocal conic that touches PQ, the tangents being reckoned positive when drawn towards the interior of the ellipse.

XIV.

1. If AEB be the chord, AD the tangent, and BD the subtense, for an arc ACB of finite curvature at A, find the limit of the ratio area $ACBE$: area $ACBD$, as B approaches A.

2. An arc of continuous curvature PQR is bisected in Q, PT is the tangent at P; prove that, ultimately, as R approaches P, the angle RPT is bisected by PQ.

3. If AB be an arc of finite curvature bisected in C, and T be a point in the tangent at A, at a finite distance from A, prove that the angle BTC will be ultimately three times the angle CTA, when B moves up to A.

4. Two curves touch one another, and both are on the same side of the common tangent. If in the plane of the curves this tangent revolve about the point of contact, or if it move parallel to itself, the prime ratio of the nascent chords in the former case will be the duplicate of their prime ratio in the latter case.

5. ACB is a small arc of finite curvature; prove that the mean of the distances of every point of the arc from the chord AB is equal to $\frac{2}{3}$ of the distance of the middle point of the arc from the chord, and that the mean of the distances of every point of the arc from the tangent at either extremity of the arc is equal to $\frac{2}{3}$ of the distance of the middle point of the arc from the same tangent.

6. When on an arc of continuous curvature there is no point where the curvature is a maximum or minimum, the circles of curvature at the extremities of the arc cannot intersect.

7. If S be any point in the plane of a curve, P any point on the curve, Y the corresponding point on the pedal for which S is the pole, V the point where PS cuts the circle of curvature at P, V' the corresponding point for the pedal, then $4SP.SV' = PV.YV'$.

8. The radius of curvature in a curve increases uniformly with its inclination to a fixed radius. Prove that the area between the curve, its evolute, and the two radii of curvature of lengths a, b, which contain an angle ϕ, is $\frac{1}{6}(a^2 + ab + b^2)\phi$.

9. A curve is such that the radius vector makes half the angle with the normal that it does with a fixed line; find the chord of curvature through the pole.

10. In a segment of an arc of finite curvature a pentagon is inscribed, one side of which is the chord of the arc, and the remaining sides are equal. Shew that the limiting ratio of the areas of the pentagon and segment, when the chord moves up towards the tangent at one extremity, is 15 : 16.

11. APQ is a curve of continued and finite curvature, P and Q are two points in it, whose abscissæ along the normal at A are always in the ratio $m : 1$, and from B, C, two points in the normal, straight lines BPb, CPc, BQb', CQc' are drawn to meet the tangent at A. Shew that, when P and Q move up to A, the areas of the triangles bPc, $b'Qc'$ are ultimately in the ratio $m^{\frac{3}{2}} : 1$.

12. AB is an arc of finite curvature at A, and a point P is taken such that $AP : PB$ is in the constant ratio of $m : n$. Tangents at A and B intersect the tangent at P in T and R, and AB is joined. Prove that the ultimate ratio of the area $ATRB$ to the segment APB, as B moves up to A, is $3(m^2 + mn + n^2) : 2(m + n)^2$.

13. The tangent to a curve at a point B meets the normal at a point A in T, C is the centre of curvature at A, and O a point on AC; prove that, in the limit, when B moves up to A, the difference of OA and OB bears to AT the ratio $OC : OA$.

14. O is a point within a closed oval curve, P any point on the curve, QPQ' a straight line drawn in a given direction, such that $QP = PQ' = PO$; prove that, as P moves round the curve, Q, Q' trace out two closed loops, the sum of whose areas is twice the area of the original curve.

NOTE ON MAXIMA AND MINIMA.

93. When a variable magnitude changes its value in consequence of the change of some element of its construction, the law of its variation can be graphically represented by the form of a curve in which the ordinate and abscissa of every point represent respectively the corresponding values of the variable magnitude and of the element on which it depends.

Examples of this mode of representation have been given in Arts. 55 and 57, in which the time or space is the element upon which depends the velocity or kinetic energy, which are the variable magnitudes respectively considered.

94. This graphic representation enables us to obtain a property of any maximum or minimum value of a variable magnitude which is applicable to the solution of a variety of problems.

For, let Ox be the line of abscissæ and B a point in the auxiliary curve at which the tangent RBS to the curve is parallel to Ox, and let the abscissa OA represent the corresponding value of the element, then the ordinate AB is a maximum or minimum according as the portion of the curve PBQ in the neighbourhood of B is concave or convex to the line Ox.

Let a chord PQ be drawn parallel to the tangent RBS, the two points P and Q one on each side of B have equal ordinates MP, NQ, which, as PQ moves up to and continues parallel to the tangent, become nearer and nearer and are ultimately equal to the maximum or minimum value, while the difference between the corresponding abscissæ ultimately vanishes.

Hence is derived the following theorem:

If a variable magnitude have a maximum or minimum value there will be two values of the element of construction, one greater and the other less than the critical value, which will give equal values of the variable magnitude.

95. *Stationary value of a magnitude.*

Let the equal ordinates MP, NQ be produced to meet the tangent in R and S, then by Lemma XI., PR and QS vanish compared with AM or AN, and the ratio of the rates of increase of the ordinate to that of the abscissa, which is generally finite, vanishes for the critical case of a maximum or minimum; on this account the magnitude is said to have a *stationary* value.

One or two examples are sufficient to shew the application of this method.

96. *To find at what point on the bank of an oval pond a person must land in order to pass from a given point on the pond to a given point on the bank in the shortest possible time, having given the ratio of his rates by land and by water.*

Let A, B be the two given points, P the point at which he must land, and let nv, v be the velocities by water and along the bank. On opposite sides of P there are two points Q, R at which if he land the time to B will be the same, in AR take $AM = AQ$, then MR in water and QR on land are described in the same time, therefore $n \cdot QR = MR$, which is true, however near Q and R may be to P; therefore $\cos\phi = n$, where ϕ is the angle between AP and the tangent at P; whence, when the exact form of the oval is given, the position of P can be found.

97. *To find the chord of an oval, which, drawn through a given point, cuts off a maximum or minimum segment.*

Through the fixed point A it is possible to draw two chords PAQ and pAq, one on each side of the required chord, for which the areas cut off are exactly equal; take away the common part, and the remainders PAp, QAq are equal; therefore, ultimately, when the angle between them vanishes, $PA \cdot pA = QA \cdot qA$, and the chord which cuts off a maximum or minimum area must be bisected by the fixed point.

98. *If a triangle of constant shape be described about a given triangle, prove that when the area is a maximum the normals to*

the sides of the circumscribed triangle at the angular points of the given triangle will meet in a point.

Let ABC be the given triangle, $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ two positions of the circumscribing triangle whose areas are equal, the triangle of maximum area being intermediate in position.

Since the angles at α, α' are equal, the points α, α' lie in the same segment of a circle whose base in BC, and the angles $\alpha C\alpha'$, $\alpha B\alpha'$ are equal. Hence the triangles $\alpha C\alpha'$, $\beta C\beta'$, $\beta A\beta'$, $\gamma A\gamma'$, &c., are ultimately proportional to $C\alpha^2$, $C\beta^2$, $\ldots$.

But the sum of the areas $\alpha C\alpha'$, $\beta A\beta'$, $\gamma B\gamma'$ are ultimately equal to the sum of $\beta C\beta'$, $\gamma A\gamma'$, $\alpha B\alpha'$,

$$\therefore \ \alpha C^2 - \beta C^2 + \beta A^2 - \gamma A^2 + \gamma B^2 - \alpha B^2 = 0.$$

Let the normals at A and C meet in N;

$$\therefore \ \alpha C^2 - \beta C^2 = \alpha N^2 - \beta N^2,$$
$$\beta A^2 - \gamma A^2 = \beta N^2 - \gamma N^2;$$
$$\therefore \ \alpha B^2 - \gamma \beta^2 = \alpha N^2 - \gamma N^2 = \alpha D^2 - \gamma D^2,$$

if ND be perpendicular to $\alpha\gamma$;

$$\therefore \ \alpha B - \gamma B = \alpha D - \gamma D ; \ \ \therefore \ BD = 0,$$

which proves the proposition.

XV.

1. In an arc AB of continuous curvature n points P_1, P_2, .. are taken so that the polygon $AP_1P_2..B$ has a maximum area; prove that, when the arc AB is indefinitely diminished, the arcs AP_1, P_1P_2, .. are all equal.

2. Find the greatest rectangle which can be inscribed in a triangle, one side of which is on a side of the triangle.

3. Prove that the diagonals of the greatest rectangle which can be inscribed in an ellipse, having its sides parallel to the axes, are the equi-conjugate diameters.

4. Prove that the parallelograms of smallest area which can be described about a given ellipse are those which have their sides parallel to conjugate diameters.

5. A point O is taken on the major axis AA' of an ellipse produced, and a line is drawn through O cutting the ellipse in the points P and P'. Prove that when the area of the quadrilateral $APP'A'$ is a maximum the projection of PP' upon AA' is equal to the semi-axis-major.

6. Prove that the quadrilateral of maximum area that can be formed with four straight lines AB, BC, CD, DA, of given lengths is such that a circle can be described about it. Hence prove that the curve of given length which on a given chord encloses a maximum area is an arc of a circle.

7. From a point T on the exterior of two oval curves tangents TP, TQ are drawn to the inner; shew that, when the arc PQ is a minimum or maximum, the radii of curvature at P and Q are in the ratio $TP \sec\alpha : TQ \sec\beta$, where α, β are the angles which TP, TQ respectively make with the normal at T.

8. Find the ultimate intersection of the chords common to an ellipse and two consecutive circles of curvature, and shew that when the common chord attains its maximum length for a given ellipse, it cuts the ellipse at angles whose tangents are as $1 : 3$.

9. A triangle inscribed in a closed oval curve moves so that two of its sides cut off constant areas. Prove that when the area cut off by the third side is stationary the three lines formed by joining each angular point of the triangle to the intersection of tangents at the other two points are concurrent.

10. Any two normal chords of an ellipse at right angles to each other cut off equal areas from the curve. Hence find the position of the normal chord which cuts off the minimum area.

11. An endless string just reaches round the circumference of an oval, and when it is cut at any point it is unwrapped until it becomes a tangent at the point of section; shew that the involute so formed will have a maximum or minimum length if the point of section be chosen so that the length of the oval shall be equal to the circumference of the circle of curvature at that point.

DIGRESSION

ON THE PROPERTIES OF CERTAIN CURVES.

THE CYCLOID.

99. DEF. If, in one plane, a circle roll along a straight line, any point on its circumference will describe a curve called a *Cycloid.*

Let C, D be the points where the tracing point P meets the straight line, on which it rolls; A the point where it is furthest from CD, AB being the corresponding diameter of the circle.

The rolling circle is called the *generating circle*, AB is called the *axis*, A the *vertex*, CD the *base*, and C, D the *cusps.*

100. Let RPS be the generating circle in any position, then, since the points of the base and circle come successively in contact without slipping, $CS = $ arc PS, CB and BD are each half of the circumference of the circle, and $BS = $ arc RP.

101. *To draw a tangent to a cycloid.*

Let the generating circle be in the position RPS, then, considering a circle as the limit of a regular polygon of a large number of sides, it will roll by turning about the point of contact, which will be at rest for an instant, being an angular point of the polygon; therefore for an instant P will move perpendicular to SP, or in the direction PR of the supplemental chord, which will therefore be the tangent to the cycloid at P.

If AQB be the circle on AB as diameter, PQM an ordinate perpendicular to AB, the tangent at P will be parallel to the chord QA.

102. *To find the length of the arc of a cycloid.*

Let RPS be the position of the generating circle corresponding to the point P in the cycloid, PR being the tangent at P.

When the circle has turned through any angle POp the centre O will have moved through a distance equal to Pp, and the motion of the generating point will be the resultant
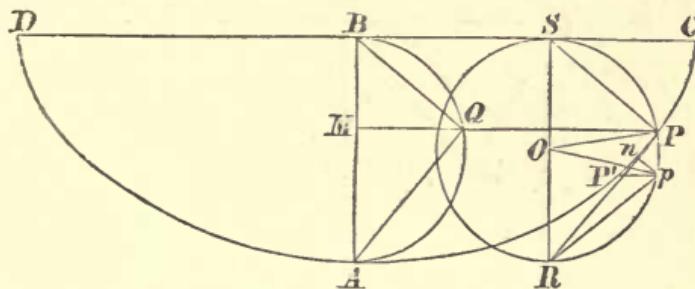
of Pp due to the rotation, and $pP' = Pp$ parallel to the base due to the translation of the centre of the circle; and PP will ultimately coincide with PR. Draw pn perpendicular to PR, then, since $Pp = P'p$, $PP' = 2Pn = 2(RP - Rp)$ ultimately. Hence the arc of the cycloid measured from the vertex increases twice as fast as the chord of the generating circle, which is a tangent to the cycloid, and they vanish simultaneously, therefore the arc of the cycloid is double of the chord of the generating circle, or referring to the circle on the axis AB as diameter, the arc AP is double of the corresponding chord AQ.

103. *To find the relation between the arc and abscissa.*

Let AM be the abscissa of the point P, then

$$AM : AQ :: AQ : AB;$$
$$\therefore AP^2 = 4AQ^2 = 4AB \cdot AM.$$

104. *To shew that the evolute of a given cycloid is an equal cycloid, and that the radius of curvature of a cycloid is twice the normal.*

Let APC be half the given cycloid, AB the axis, A the vertex, and BC the base. Produce AB to C', making BC' equal to AB, and complete the rectangle $BCB'C'$, and let the semi-cycloid $C'P'C$ be generated by a circle, whose diameter is equal to that of the generating circle of the given cycloid, rolling on $C'B'$; C being the vertex, CB' the axis of this cycloid.

Let SPR, SPR' be two positions of the respective generating circles, having their diameters RS, SR' in the same

straight line, P, P' being the corresponding points of the cycloids; join SP, PR and SP', $P'R'$.
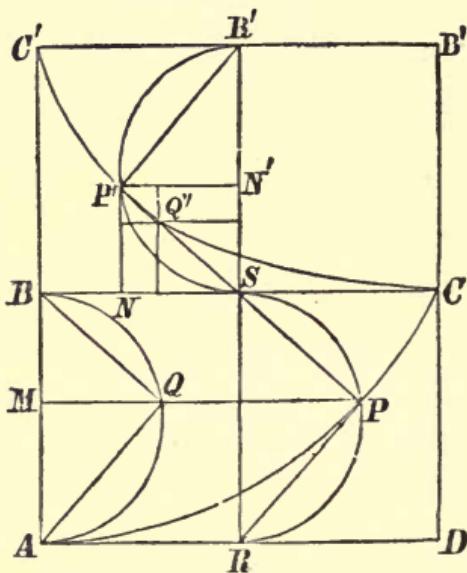
By the mode of generation, arc $SP = SC$, and arc $SPR = BC$;

$$\therefore \text{ arc } PR = BS = C'R' = \text{arc } P'R';$$

$$\therefore \angle PSR = \angle P'SR'; \text{ and } PSP' \text{ is a straight line.}$$

Also, arc $P'S = \text{arc } PS$; $\therefore$ chd. $P'S = \text{chd. } PS$;

$$\therefore P'SP = 2P'S = P'C \text{ the cycloidal arc;}$$

also $P'SP$ touches the cycloid $C'P'C$ at P';

therefore, a string fixed at C', and wrapped over the arc of the semicycloid, will, when unwrapped, have its extremity in the arc of the given cycloid; hence, the evolute of a semicycloid is an equal semicycloid, and the radius of curvature at P is $2PS$ or twice the normal. If another equal semicycloid be described by the circle rolling on $B'C'$ produced, the extremity of the string wrapped on this curve will trace out the remainder of the given cycloid.

Thus a pendulum may be made to oscillate in a given cycloid.

105. *To find the area of the cycloid.*

Let P, P' be two points very near each other in a cycloid,

Q, Q' corresponding points in the generating circle, p, p' in the evolute, R, R' the intersections of the base with normals Pp,
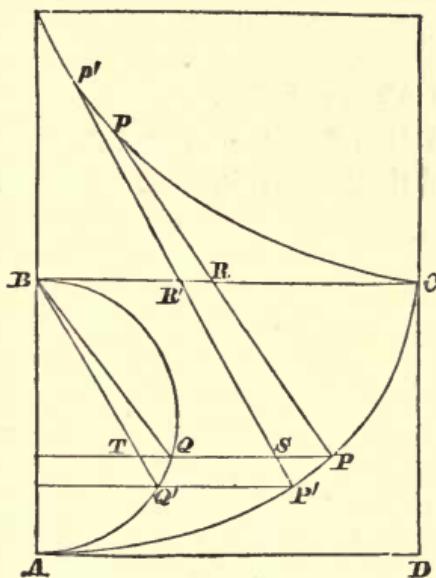
$P'p'$, T, S the intersections of BQ' and $P'p'$ with PQ. Then $pR = PR = BQ$, and $\triangle p'PS = 4\triangle p'RR'$ ultimately $= 4\triangle BQT$; therefore trapezium $PRR'S = 3\triangle BQT$ ultimately, and the same being true for all the inscribed triangles and trapeziums, whose sums are ultimately the areas of the semicircle and semicycloid, therefore, by Cor., Lemma IV., the area of the cycloid is three times that of the generating circle.

106. The following method of finding the area of a cycloid is independent of the properties of the evolute.

In the figure of Art. 104 let P' be any point in the cycloid $CP'C'$, $P'S$ the chord of the generating circle which touches the cycloid, and let Q' be a point in the cycloid near P', then the arc $P'Q'$ ultimately coincides with $P'S$. Let $Q'N'$, $Q'N$ be the complements of the parallelogram whose diagonal is $P'S$, and sides parallel and perpendicular to the base, these are equal ultimately; therefore, by Lemma IV., the cycloidal area $CNP' = $ circular segment $SP'N'$.

The exterior portion $CB'C'$ is equal to the area of the semicircle, and the whole parallelogram $BCB'C'$ is the rectangle

under the diameter and semi-circumference of the generating circle, and is equal to four times the area of the semicircle; therefore the cycloidal area $CC'B'$ is three times the area of the semicircle.

107. *All cycloids are similar.*

Let two cycloids APC, Apc be placed so that their vertices are the same, and their axes coincident in direction, and describe
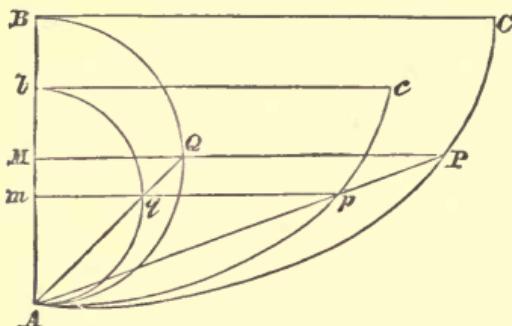
circles on the axes AB, Ab as diameters. Draw AqQ cutting the circles in q, Q. Then, since the segments Aq, AQ are similar, arc Aq : arc AQ :: Aq : AQ; and, if mqp, MQP be ordinates to the cycloids, arcs Aq, $AQ = qp$, QP respectively; therefore qp : QP :: Aq : AQ, and ApP is a straight line. Also Ap : AP :: Aq : AQ :: Ab : AB, a constant ratio; hence the cycloids satisfy the condition of similarity, and in this position of the cycloids the point A is a centre of direct similitude.

108. *To construct a cycloid which shall have its vertex at a given point, its base parallel to a given straight line, and which shall pass through a given point.*

Let A be the given vertex, AB perpendicular to the given line, P the given point. In AB take any point b, and with the generating circle, whose diameter is Ab, describe a cycloid Apc, join AP intersecting this cycloid in p.

Take AB a fourth proportional to Ap, AP, and Ab; then AB will be the diameter of the generating circle of the required cycloid; for, since Ap : AP :: Ab : AB, and all cycloids are similar, P is a point in the cycloid whose axis is AB.

109. *A particle slides down the smooth arc of a cycloid,* *whose axis is vertical, and vertex downwards, to find the time* *of an oscillation.*

Let AB be the vertical axis of the cycloidal arc APL, L the point from which the particle begins to move, PQ a small arc of
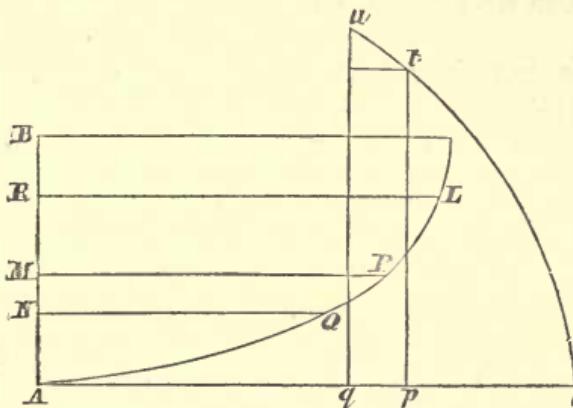
its path, LR, PM, QN perpendicular to AB; and take Al, Ap, Aq on the tangent at A respectively equal to AL, AP, AQ.

Suppose a point to move from l to A in the same time as the particle moves on the cycloid from L to A, their velocities being always equal at equal distances from A.

Let v be the velocity at P or p, and T the time of falling from B to A, so that $v^2 = 2gRM$ and $2AB = gT^2$; therefore $v^2T^2 = 4AB \cdot RM = 4AB \cdot AR - 4AB \cdot AM = AL^2 - AP^2$, Art. 103, $= Al^2 - Ap^2$.

Describe a circle with centre A and radius Al, and draw the ordinates pt, qu, then $Al^2 - Ap^2 = pt^2$, and $pt = vT$; and if τ be the time from P to Q, $PQ = pq = v\tau$ ultimately, hence

$$tu : Al :: pq : pt :: \tau : T;$$

therefore, if a point move in the circle from l with uniform velocity $\dfrac{Al}{T}$, the point moving in lA will always be in the foot of the ordinate and the motion in lA or LA will therefore be a simple harmonic motion, by (5) page 78.

The time from L to A is the time of describing the quadrant $\frac{1}{2}\pi Al$ with velocity $\dfrac{Al}{T}$, $= \frac{1}{2}\pi T = \frac{1}{2}\pi \sqrt{\dfrac{2AB}{g}}$.

The length of the string which, by the contrivance of Art. 104, makes a particle oscillate in this cycloid is $2AB=l$ suppose; therefore the time of the oscillation of a cycloidal pendulum of length l from rest to rest $= \pi \sqrt{\dfrac{l}{g}}$.

NOTE. The time from L to P is $\sqrt{\dfrac{2AB}{g}} \times \cos^{-1} \dfrac{AP}{AL}$.

110. We can shew that the motion on the cycloid is a simple harmonic motion by the first definition, (5) page 78; for, referring to the figure, page 115, since the tangent at P is parallel to AQ, the acceleration along the curve at P is $g \cdot \dfrac{AQ}{AB} = g \cdot \dfrac{AP}{2AB}$, which varies as AP, and, by (4) page 77, the time from L to A is obtained.

111. *To find the time of a very small oscillation of a simple pendulum suspended from a point.*

A *simple pendulum* is an imaginary pendulum consisting of a heavy particle called the *bob*, suspended from a *point* by means of a rod or string without weight.

In this case the pendulum describes the small arc of a circle which may be considered the same as a cycloidal arc, the axis of which is half the length l of the pendulum, therefore the time of oscillation from rest to rest is $\pi \sqrt{\dfrac{l}{g}}$.

112. *To count the number of oscillations made by a given pendulum in any long time.*

In consequence of the liability to error in counting a very great number of oscillations, since in the case of a seconds pendulum there would be 3600 oscillations for each hour, it becomes necessary to adopt some contrivance for diminishing the labour. For this purpose the pendulum is made to oscillate nearly in the same time as that of a clock; it is then placed in front of that of the clock, so that when they are simultaneously near their lowest positions the bob of the pendulum and a cross marked on the pendulum of the clock may be in the field of view of a fixed telescope.

Suppose that after n oscillations of the given pendulum they are again in coincidence close to the same position; if there be m such coincidences in the whole time of observation, the number of oscillations in that time will be mn; thus the only labour has been to count the n oscillations, and to estimate the number of the coincidences before the last one observed.

113. *To measure the accelerating effect of gravity by means of a pendulum.*

Let g be the measure of this effect or the velocity generated by the force of gravity in a second.

Let l be the length of a simple pendulum which makes n oscillations in m hours, then $\dfrac{3600m}{n}$ = number of seconds in one oscillation $= \pi \sqrt{\dfrac{l}{g}}$; therefore $g = \dfrac{\pi^2 l n^2}{(3600)^2 m^2}$, in whatever unit of length l is estimated.

This would be a very exact method of determining g, if we could form a simple pendulum; but it is impossible to do this, and it is only by calculations of a nature too difficult to be explained here that it can be shewn how to deduce the length of the simple pendulum, which would oscillate in the same time as a pendulum of a more complicated structure.

114. The seconds pendulum at any place is the simple pendulum which at the mean level of the sea at that place would oscillate in one second.

If L be the length of the seconds pendulum, l the length of a pendulum making n oscillations in m hours,

$$\therefore \ \pi \sqrt{\frac{l}{g}} = \frac{3600m}{n}, \text{ and } \pi \sqrt{\frac{L}{g}} = 1, \ \therefore \ L = \frac{n^2 l}{(60)^4 m^2}.$$

115. *To determine the height of a mountain by means of a seconds pendulum, the force of gravity at any point being supposed to vary inversely as the square of the distance from the centre of the earth.*

Let L be the length of a seconds pendulum, x the height of the mountain above the mean level of the sea, a the radius

R

of the earth, all expressed in feet; and let n be the number of seconds lost in 24 hours by the pendulum at the top of the mountain.

If g be the measure of the accelerating effect of gravity at the mean level of the sea, then $\dfrac{ga^2}{(a+x)^2}$ will be its value at the top of the mountain, and the time of oscillation at the top will be $\pi\sqrt{\left\{\dfrac{L}{g}\left(\dfrac{a+x}{a}\right)^2\right\}}$, or $\dfrac{a+x}{a}$ seconds, since $\pi\sqrt{\dfrac{L}{g}}=1$;

hence, writing N for $24\times60\times60$, $(N-n)\dfrac{a+x}{a}=N$, and

$1+\dfrac{x}{a}=\dfrac{N}{N-n}=1+\dfrac{n}{N}+\dfrac{n^2}{N^2}\cdots$; $\therefore\ x=\dfrac{a}{N}n+\dfrac{a}{N^2}n^2$, nearly, but $a=4000\times1760\times3$ and $N=24\times60\times60$, therefore the height of the mountain will be $244\dot{\cdot}4n+\cdot0027n^2$; thus, if $n=10$, the height will be $2444\cdot7$ feet.

NOTE. The attraction of the mountain would make a sensible variation from the law of the inverse square, this law being true only if the earth consisted of homogeneous spherical strata.

116. *To find the number of seconds lost in a day, in consequence of a slight error in the length of the seconds pendulum; and conversely.*

Let N be the number of seconds in a day, L the length of the seconds pendulum, $L+\lambda$ that of the incorrect pendulum, $N-n$ the number of its oscillations in a day;

$$\therefore\ (N-n)\,\pi\sqrt{\dfrac{L+\lambda}{g}}=N.\pi\sqrt{\dfrac{L}{g}};\ \ \therefore\ \dfrac{\lambda}{L}=\dfrac{2n}{N}\ \text{nearly.}$$

THE EPICYCLOID AND HYPOCYCLOID.

117. DEF. The curve traced out by a point on the circumference of a circle, which rolls upon that of a fixed circle, is called an *Epicycloid* if the concavities of the two circles be in opposite directions, a *Hypocycloid* if the concavities be in the same direction.

118. *To shew that the evolute of an epicycloid is a similar epicycloid.*

Let FA be the fixed circle, APE the rolling circle in any position, P the generating point, CAE a line drawn through
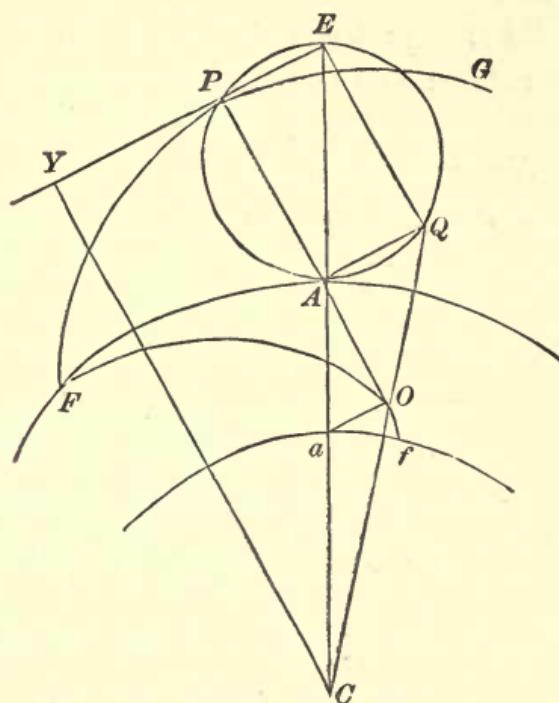
the point of contact, meeting the rolling circle in A, E; and let GPF be the epicycloid, of which PA and PE will be a normal and tangent.

Draw the chord EQ parallel to PA and join CQ meeting PA produced in O. Since EQ is parallel to AO,

$$CO : CQ :: CA : CE;$$

therefore O and Q describe similar figures. But Q, being the other extremity of the diameter through P, will describe an epicycloid similar and equal to GPF, being at its cusp when P is at G the greatest distance from C.

Draw Oa parallel to QA and therefore perpendicular to PO, meeting CA in a, then O generates an epicycloid fF by the rolling of a circle AOa, whose diameter is Aa, on a fixed circle of radius Ca.

Also PO the normal to GF is perpendicular to aO and is therefore a tangent to fF, hence fF is the evolute of the given epicycloid and is a similar epicycloid.

Let a, b be the radii of the fixed and rolling circles for the given epicycloid, then

$$Aa : CA :: OQ : CQ :: AE : CE :: 2b : a + 2b;$$

therefore $Aa : AE :: a : a + 2b$, and if $a = \infty$, $Aa = AE$, and AF, af become straight lines, whence the evolute of a cycloid is an equal cycloid.

119. Since $AO : PA :: AO : EQ :: CA : CE$, therefore $PO : PA :: 2(a + b) : a + 2b$, which gives PO the radius of curvature at P of the given epicycloid; this will be found independently of the evolute in Art. 121 below.

120. *To find the length of any arc of the epicycloid.*

By the properties of the evolute, see the last figure, the arc OF of the evolute $= OP = 2AP \cdot \dfrac{a+b}{a+2b}$, and the arc of the epicycloid generated by Q, measured from Q to the highest point, $= OF \dfrac{a+2b}{a} = 2AP \cdot \dfrac{a+b}{a}$; therefore the arc GP from the highest point G of the epicycloid $GPF = 2EP \cdot \dfrac{a+b}{a}$.

121. *To find the radius of curvature at any point of an epicycloid.*
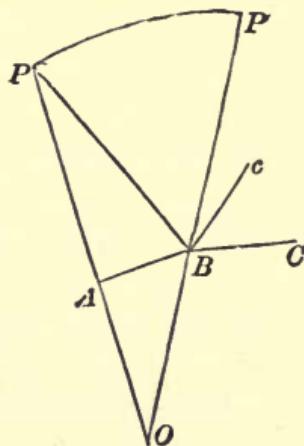
Let AB, BC be consecutive sides of a fixed regular polygon of m sides, AB, Bc sides of another regular polygon of n sides equal to those of the former, on the outside of which it rolls, in a position in which two sides are coincident.

Let P be any angular point of the rolling polygon; P will generate a figure composed of a series of circular arcs such as PP', P' being the position of P when Bc coincides with BC. Produce PA, $P'B$ to meet in O.

Then $\angle APB = \dfrac{\pi}{n}$, and $\angle PBP' = \angle cBC = \dfrac{2\pi}{m} + \dfrac{2\pi}{n}$;

$$\therefore \angle POB = 2\pi \left(\frac{1}{m} + \frac{1}{n}\right) - \frac{\pi}{n} = \pi \left(\frac{2}{m} + \frac{1}{n}\right);$$

$$\therefore PO : PB :: \sin 2\pi \left(\frac{1}{m} + \frac{1}{n}\right) : \sin \pi \left(\frac{2}{m} + \frac{1}{n}\right).$$

When the number of sides is indefinitely increased, the polygons ultimately become circles, the curve traced out by P becomes an epicycloid, and PO the radius of curvature at P.

If a, b be the radii of the fixed and rolling circles $m.AB = 2\pi a$ and $n.AB = 2\pi b$, ultimately; therefore $m : n :: a : b$;

$$\therefore PO : PA :: 2\left(\frac{1}{m} + \frac{1}{n}\right) : \frac{2}{m} + \frac{1}{n} :: 2(a+b) : a+2b;$$

therefore the radius of curvature is $2PA \cdot \dfrac{a+b}{a+2b}$, where PA is the part of the normal intercepted between the generating point and the point of contact.

If $a = \infty$, or the fixed circle become a straight line, the epicycloid will become a cycloid, and the radius of curvature will be twice the normal, as in Art. 104.

122. *To find the area of an epicycloid.*

In the last figure, area $APP'B = \triangle PAB + \text{sector } PBP'$; now

$$\text{sector } PBP' = \tfrac{1}{2}PB^2 \cdot 2\pi \left(\frac{1}{m} + \frac{1}{n}\right) \text{ and } \triangle PAB = \tfrac{1}{2}PB^2 \sin \frac{\pi}{n};$$

$$\therefore \text{ area } APP'B = \triangle PAB \left\{1 + \frac{2(m+n)}{m}\right\} \text{ ultimately;}$$

hence, by Lemma IV. Cor., the area of the segment of the

epicycloid included between two normals and the fixed circle is $\left(3 + \dfrac{2b}{a}\right) \times$ the corresponding segment of the rolling circle. Compare Art. 105.

123. The corresponding properties of the hypocycloid may be proved in a similar manner; and the results obtained will be the same as for the epicycloid, if in the latter the sign of b be changed.

Thus, if the diameter of the fixed be double that of the rolling circle, the hypocycloid will become a straight line, which agrees with the result of Art. 121, since $a + 2b = 0$, and therefore the radius of curvature at every point will be infinite.

THE EQUIANGULAR SPIRAL.

124. DEF. The *equiangular spiral* is a curve which cuts all the radii drawn from a fixed point at a constant angle.

125. If a series of radii SA, SB, SC, ... be drawn inclined at equal angles, and AB, BC, CD, ... make equal angles SAB, SBC, ... with these radii respectively, the curvilinear limit
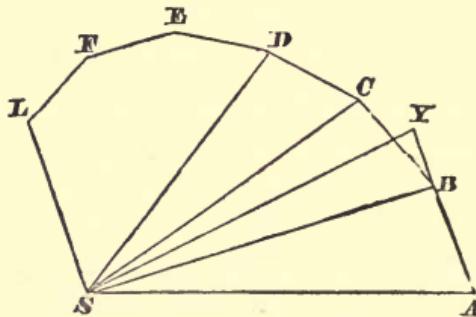
of the polygon $ABCD$ ..., when the equal angles ASB, BSC, ... are indefinitely diminished, will be an equiangular spiral.

126. *To find the length of an arc of an equiangular spiral contained between two radii.*

Let α be the constant angle SAB, and let SL be the nth

radius from SA; then, since the triangles ASB, BSC, ... are similar, $SA : SB :: SB : SC$ ....

Let $SB = \lambda . SA$, then $BC = \lambda . AB$, $CD = \lambda^2 . AB ... FL = \lambda^{n-1} . AB$;

$\therefore AB + BC + ... + FL : AB :: 1 + \lambda + ... + \lambda^{n-1} : 1 :: 1 - \lambda^n : 1 - \lambda$

$:: SA - \lambda^n . SA : SA - SB :: SA - SL : SA - SB,$

but $AB \cos \alpha = SA - SB \cos ASB = SA - SB$ ultimately, and $AB + BC + ...$ is ultimately the arc of the spiral; therefore $\text{arc } AL = (SA - SL) \sec \alpha.$

127. *To find the area of an equiangular spiral bounded by two radii.*

Employing the same construction as above,

$\triangle ASB + \triangle BSC + \triangle CSD + ... : \triangle ASB :: 1 + \lambda^2 ... + \lambda^{2n-2} : 1$

$:: 1 - \lambda^{2n} : 1 - \lambda^2 :: SA^2 - SL^2 : SA^2 - SB^2,$

but $SB^2 = SA^2 - 2SA . AB \cos \alpha + AB^2$ and $\triangle ASB = \frac{1}{2} SA . AB \sin \alpha$;

$\therefore SA^2 - SB^2 = 4 \triangle ASB \times \cot \alpha$, ultimately;

$\therefore \text{area } ASL = \frac{1}{4} (SA^2 - SL^2) \tan \alpha.$

128. *To find the radius and chord of curvature through the pole at any point of an equiangular spiral.*

Let SP, SQ be radii drawn to two points P and Q, near to
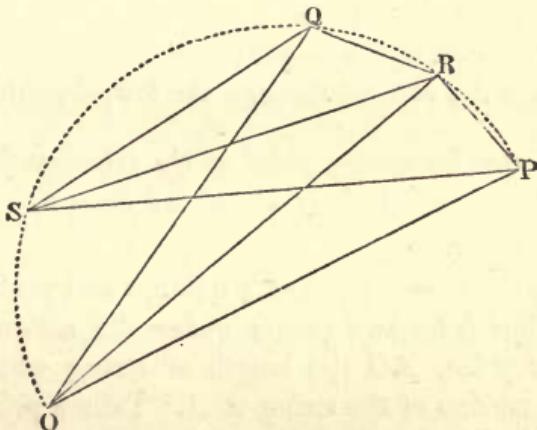
one another, let PR, QR, tangents to the spiral at P and Q, intersect in R, and let the normals PO, QO intersect in O; join OR, SR.

Then, since angles SQR, SPR are equal to two right angles, and each of the angles OQR, OPR is a right angle, the circle which passes through P, R, and Q will also pass through S and O, and OR will be its diameter; therefore $\angle OSR$ is a right angle. Hence, proceeding to the limit, O is the centre of the circle of curvature at P, and OSP is a right angle. Therefore if α be the angle of the spiral, $OP = SP \operatorname{cosec} \alpha$ will be the radius of curvature, and $2SP$ the chord of curvature through the pole.

129. The following is an illustration of Art. 86.
If PV be the chord of curvature through S,

$$SY' - SY : SP' - SP :: 2SY : PV;$$

but in the equiangular spiral $SY : SY' :: SP : SP'$;

$$\therefore \ SY' - SY : SP' - SP :: SY : SP; \text{ whence } PV = 2SP.$$

THE CATENARY.

130. DEF. The *Catenary* is the curve in which a uniform and perfectly flexible string, of which the extremities are suspended at two points, would hang under the action of gravity, supposed to be a constant force acting in parallel lines.

The *directrix* is a horizontal straight line whose depth below the lowest point is equal to the length of string whose weight is equal to the tension at the lowest point.

The *axis* is the vertical through the lowest point.

131. *The tension at any point of the catenary is equal to the weight of the string which, if suspended from that point, would extend to the directrix.*

Let A be the lowest point of a uniform and perfectly flexible string hanging from two points under the action of gravity, P any other point, AO the length of string whose weight is equal to the tension of the string at A. Take a point B in OA, and let OM, BC drawn horizontally meet a vertical PM in M and C.

If a string pass round smooth pegs at $APCB$, it is evident that there will be a position of equilibrium whatever be the

length of the string, or the position of BC, and for some length and some position of BC the tangent at A will be horizontal.
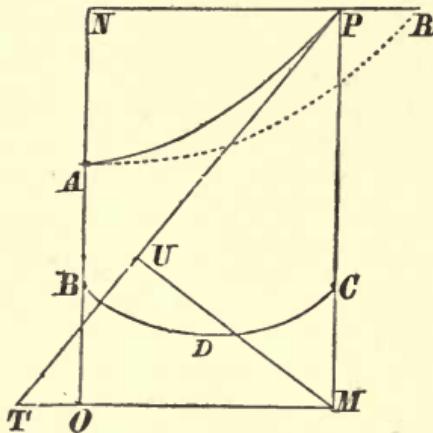
Also, since BDC will hang symmetrically, the tensions of the string at B and C will be equal, and BDC may be removed and replaced by equal lengths BO, CM of the string, without disturbing the equilibrium of AP, therefore the tension of the catenary at P is equal to the weight of a string of length PM.

132. The proposition of the preceding article may be proved by considering the catenary as the limit of the polygon formed by a series of equal rods of the same substance jointed freely at the extremities and suspended from two fixed points, when the length of the rods is indefinitely diminished.
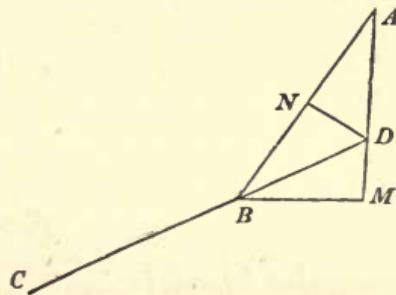
The equilibrium will be undisturbed if each rod be replaced by two weights at the extremities, each equal to half that of the rod, connected by a rod without weight.

Let AB, BC be two consecutive positions of the rods,

S

weights equal to those of the rods being placed at A, B, C; let AM be vertical and BM horizontal, and produce CB to meet AM in D; draw DN perpendicular to AB.

The forces which keep B in equilibrium act in the directions of the sides of the triangle ABD, and are proportional to them.

Therefore the difference of the tensions of AB and BC is to the weight of the rod AB as $AB - BD : AD$, that is, ultimately, as $AN : AD$ or $AM : AB$; hence the difference of the tensions is the weight of a rod of length AM.

Therefore, proceeding to the limit, the difference of tensions at any two points of the catenary is equal to the weight of string, which is equal in length to the vertical depth of one point below the other, whence the truth of the proposition.

133. *P is a point in a catenary, PM perpendicular to the directrix, PT a tangent at P, MU perpendicular to PT; to shew that PU is equal to the arc measured from the lowest point, and that MU is constant.*

Let PT, fig. for Art. 131, meet the direction OM in T, and let AO be the axis, then since the arc AP supposed to become rigid is in equilibrium under the action of the tensions at A and P and the weight, and these forces are in the directions of the sides of the triangle TPM,

$$\therefore AP : AO : PM :: PM : MT : TP :: PU : MU : PM,$$

by similar triangles TPM, MPU;

$$\therefore PU = AP \text{ and } MU = AO.$$

134. *To draw a tangent to a catenary at any point.*

With centre M and radius equal to AO describe a circle, and draw PU touching this circle in U; then, since MU, which is perpendicular to PU, is equal to AO, PU will be the tangent at P.

135. *If a rectangular hyperbola be described, having centre O and semi-transverse axis OA, the ordinate of the hyperbola will be equal to the arc of the catenary.*

For, let AR be the hyperbola, therefore

$$RN^2 = ON^2 - OA^2 = PM^2 - UM^2 = PU^2; \ \therefore RN = PU = AP.$$

136. *To find the radius and vertical chord of curvature of a catenary.*

Let PQ be a small arc of a catenary, $RSPT$, QS tangents at P and Q, PM, QN ordinates, TOM the directrix.
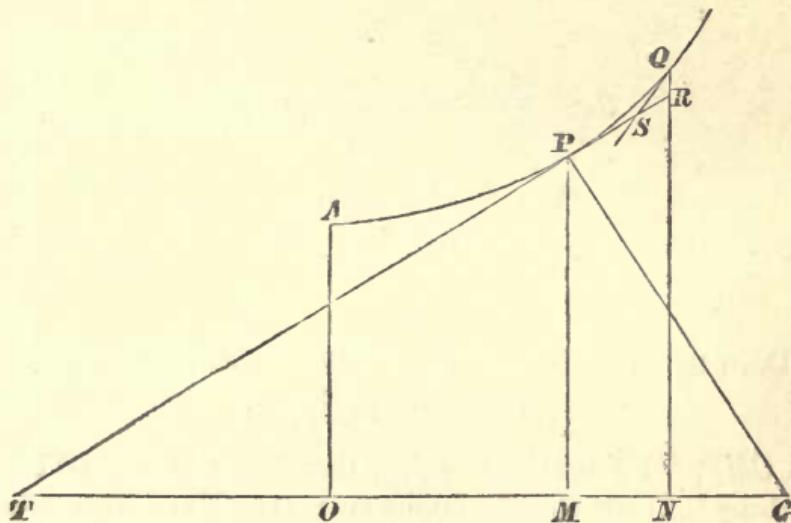
Since QRS is a triangle of the forces acting upon PQ,

tension at P: weight of PQ :: RS : QR,

$\therefore PM : PQ :: RS : QR :: \frac{1}{2}PQ : QR$, ultimately;

therefore $2PM$ is the vertical chord of curvature, and PG, the part of the normal intercepted between the point P and the directrix is equal to the radius of curvature at P.

Also $PG : PM :: PT : TM ::$ tension at P : tension at A $:: PM : AO$, therefore the radius of curvature is a third proportional to AO and PM.

THE LEMNISCATE.

137. DEF. The *Lemniscate* is the locus of the feet of the perpendiculars drawn from the centre of a rectangular hyperbola upon the tangent.

138. *To find the inclination to the tangent at any point of the radius from the centre of the lemniscate.*

Let CY be perpendicular on PT the tangent at the point P

in the hyperbola, then $CY . CP = PF . CD = AC^2$, since $AC = BC$ and $CP = CD$ in the rectangular hyperbola.
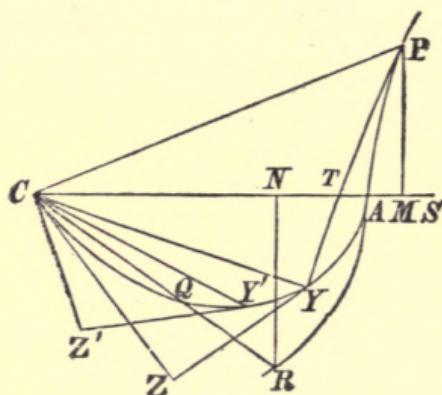
Draw the ordinate PM, then $CT . CM = AC^2 = CY . CP$;

$$\therefore\ CY : CT :: CM : CP;$$

and CMP, CYT are right angles; therefore $\angle PCM = \angle TCY$.

Draw CZ perpendicular on the tangent at Y to the lemniscate; then ZCY and YCP are similar triangles, see page 55;

$$\therefore\ \angle ZYC = \angle CPY = \text{complement of twice } \angle YCA.$$

139. *To find the perpendicular on the tangent at any point of the lemniscate.*

$$CZ . CP = CY^2, \text{ and } CY . CP = AC^2;$$

$$\therefore\ CZ : CY :: CY^2 : AC^2;$$

$$\therefore\ CZ . AC^2 = CY^3.$$

140. *To find the chord of curvature through the centre, and the radius of curvature at any point of the lemniscate.*

Let YV be the chord of curvature;

$$\therefore\ YV : 2CZ :: CY - CY' : CZ - CZ', \text{ ultimately, Art. 86,}$$

$$\text{and } (CZ - CZ')\, AC^2 = CY^3 - CY'^3,$$

$$\therefore\ CY - CY' : CZ - CZ' :: AC^2 : 3CY^2 :: CY : 3CZ;$$

$\therefore\ YV = \frac{2}{3} CY$, or the chord of curvature through the centre is two-thirds of the radius vector.

Also, the radius of curvature : $\frac{1}{2}YV$

$$:: CY : CZ :: AC^2 : CY^2 :: CP : CY,$$

hence the radius of curvature is $\frac{1}{3}CP$, or $\frac{1}{3}$ of the radius at the corresponding point of the hyperbola.

141. *Poles of the lemniscate.*

Let S, H be the foci of the hyperbola, s, h the middle points of CS and CH; s, h are called the *poles* of the lemniscate.
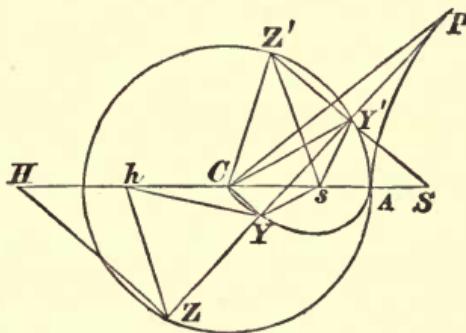
Draw SY', HZ perpendicular to the tangent to the hyperbola at P, and let SY' meet the auxiliary circle again in Z', and join sY', sZ', sY, hY, and hZ.

Since $Cs = sS$, the perpendicular from s on YY' bisects it; therefore $sY' = sY$, similarly $hY = hZ = sZ'$.

Now $SC . Ss = \frac{1}{2}SC^2 = AC^2 = SY' . SZ'$;

therefore a circle can be drawn circumscribing $CsY'Z'$; therefore $\angle Y'sZ' = \angle Y'CZ'$; also $\triangle Y'sZ' = \frac{1}{2}\triangle Y'CZ'$, since the altitude of $Y'CZ'$ is double of that of $Y'sZ'$;

$$\therefore \ sY' . sZ' = \frac{1}{2}CY' . CZ' = \frac{1}{2}CA^2;$$

therefore $sY . hY = \frac{1}{2}CA^2$, which is the property of the poles of the lemniscate.

For this proof I am indebted to Prof. Tait.

XVI.

1. If a line move parallel to the base of a cycloid, find the limit of the ratio of the segment of the cycloid to the corresponding segment of the generating circle, when the line becomes indefinitely near to the tangent at the vertex.

2. A balloon was found to be sailing steadily before the wind at an invariable elevation above the earth. A seconds pendulum suspended to the car was observed to make 2997 oscillations in 50 minutes; shew that the height of the balloon was 4 miles and 7 yards nearly, the radius of the earth being 4000 miles.

3. If a particle be made to oscillate in a cycloid on a smooth inclined plane, whose inclination to the horizon is 30°, and the base of the cycloid be horizontal, find the radius of the generating circle in order that the particle may perform a complete oscillation in n seconds.

4. If P be a point in a cycloid, and O the corresponding position of the centre of the generating circle, shew that PO will touch another cycloid of half the dimensions.

5. Shew that the limit of the whole length of an epicycloid or hypocycloid, corresponding to a complete revolution of the generating round the fixed circle, is eight times the radius of the latter, when that of the former is indefinitely diminished.

6. Prove that the epicycloid of one cusp is the pedal of a circle referred to a point in its circumference.

7. Shew that the evolute of an equiangular spiral is a similar spiral, and that the extremities of the diameters of curvature lie in a similar spiral.

8. An equiangular spiral rolls along a straight line, shew that its pole describes a straight line.

9. Prove that, if a catenary roll on a fixed straight line, its directrix will always pass through a fixed point.

10. If SY be drawn perpendicular to the tangent to a lemniscate at a point P, and SA be the greatest value of SP, prove that $SP^3 = SA^2 . SY$; S being the centre.

XVII.

1. From the consideration that the diameter of curvature is the limit of the third proportional to the subtense perpendicular to the tangent and the arc, prove that the radius of curvature of a cycloid at any point is twice the normal cut off by the base.

2. On the normal to a cycloid a constant length is measured both inwards and outwards; find the area included between the loci of the points so obtained.

3. P, Q are consecutive points on an epicycloid of two cusps; from p, q, the corresponding points of contact of the rolling with the fixed circle, pm, qn are drawn perpendicular to the cusp-line; prove that the elementary area $PQpq$ is twice the elementary area $pmnq$. Hence find the area of the epicycloid and of its evolute.

4. Prove that the diameter through the point of a rolling circle which generates an epicycloid always touches another epicycloid generated by a circle of half the dimensions.

5. A hypocycloid of n cusps has at any point a tangent drawn, prove that the length of the tangent, intercepted between the generating circle and the point of contact, is to the arc measured from the point to the vertex of the branch in which the point is taken, as $n : 2(n-1)$.

6. A bead slides on a hypocycloid being acted on by a force which varies as the distance from the centre of the hypocycloid and tending to it; prove that the time of oscillation will be independent of the arc of oscillation.

7. If, along the several normals to an epicycloid, a system of particles move from the curve under the action of a force, tending to the centre of the fixed circle, and varying as the distance, prove that they will all arrive at the fixed circle at the same instant.

8. A plane curve rolls along a straight line, shew that the radius of curvature of the path of any point, fixed with respect to the curve, is $\dfrac{r^2}{r - \rho \sin\phi}$, r being the distance of the fixed point from the point of contact, ϕ the angle between this line and the fixed line, and ρ the radius of curvature of the curve at the point of contact.

9. In an equiangular spiral, which is its own evolute, the area included between the curve and PQ, the radius of curvature at P touching the evolute in Q, is $\frac{1}{4}PQ^2 \tan a$, where a is the angle of the spiral, and PQ is supposed not to cut the curve between P and Q.

10. Prove, by the method of Lemma IV., that the area included between a catenary, the axis, the directrix, and the ordinate at any point P is twice the area of the triangle formed by the axis, the tangent at the vertex, and the straight line drawn perpendicular to the tangent at P from the point of intersection of the axis and directrix.

CENTRIPETAL FORCES.

PROP. I.   THEOREM I.

When a body revolves in an orbit, subject to the action of forces tending to a fixed point, the areas which it describes by radii drawn to the fixed centre of force, are in one fixed plane, and are proportional to the times of describing them.

Let the time be divided into equal parts, and in the first interval let the body describe the straight line
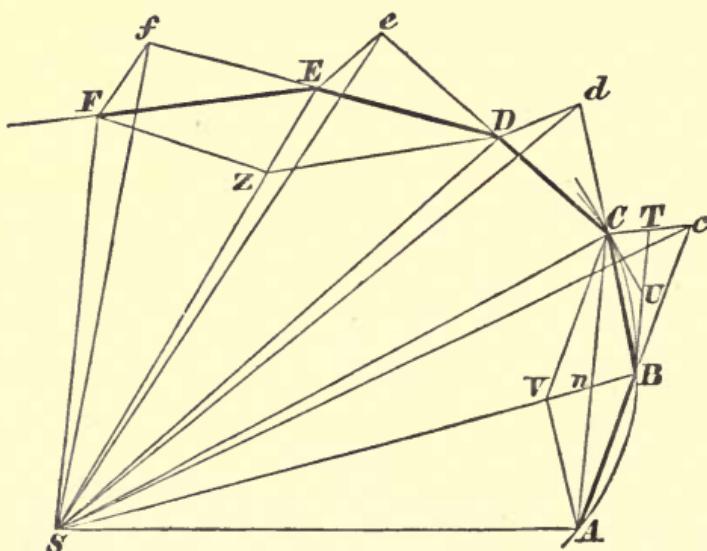
AB with uniform velocity, being acted on by no force. In the second interval it would, if no force acted, proceed to c in AB produced, describing Bc equal to AB; so that the equal areas ASB, BSc described by radii AS, BS, cS drawn to the centre S, would be completed in equal intervals.

But, when the body arrives at B, let a centripetal force tending to S act upon it by a single instanta-

neous impulse, and cause the body to deviate from the direction Bc, and to proceed in the direction BC.

Let cC be drawn parallel to BS, meeting BC in C, then, at the end of the second interval, the body will be found at C, in the same plane with the triangle ASB, in which Bc and cC are drawn.   Join SC; and the triangle SBC, between parallels SB, Cc, will be equal to the triangle SBc, and therefore also to the triangle SAB.

In like manner, if the centripetal force act upon the body successively at C, D, E, &c., causing the body to describe in the successive intervals of time the straight lines CD, DE, EF, &c., these will all lie in the same plane; and the triangle SCD will be equal to the triangle SBC, and SDE to SCD, and SEF to SDE.

Therefore equal areas are described in the same fixed plane in equal intervals; and, componendo, the sums of any number of areas $SADS$, $SAFS$, are to each other as the times of describing them.

Let now the number of these triangles be increased, and their breadth diminished indefinitely; then their perimeter ADF will be ultimately a curved line; and the instantaneous forces will become ultimately a centripetal force, by the action of which the body is continually deflected from the tangent to this curve, and which will act continuously; and the areas $SADS$, $SAFS$, being always proportional to the times of describing them, will be so in this case.   Q.E.D.

COR. 1. The velocity of a body attracted towards a fixed centre in a non-resisting medium is reciprocally proportional to the perpendicular dropped from that centre upon the tangent to the orbit.

For the velocities at the points A, B, C, D, E are as the bases AB, BC, CD, DE, EF of equal triangles, and, since the triangles are equal, these bases are reciprocally proportional to the perpendiculars from

T

S let fall upon them. [And the same is true in the limit, in which case the bases are in the direction of tangents to the curvilinear limit, therefore the velocity, &c.]

Cor. 2. If on chords AB, BC of two arcs described in equal successive times in a non-resisting medium by the same body the parallelogram $ABCV$ be completed, and the diagonal BV of this parallelogram be produced in both directions in that position which it assumes ultimately when those arcs are diminished indefinitely, it will pass through the centre of force.

Cor. 3. If, on AB, BC and on DE, EF chords of arcs described in a non-resisting medium in equal times, the parallelograms $ABCV$, $DEFZ$ be completed, the forces at B and E will be to one another in the ultimate ratio of the diagonals BV, EZ, when the arcs are indefinitely diminished.

For the velocities of the body represented by BC, EF in the polygon are compounded of the velocities represented by Bc, BV and Ef, EZ; and those represented by BV, EZ, which are equal to cC, fF, in the demonstration of the proposition were generated by the impulses of the centripetal force at B and E, and are thus proportional to those impulses. [And the same is true in the limit, in which case the ultimate ratio of the impulses at any two points is the ratio of the continuous forces at those points].

Cor. 4. The forces by which any bodies moving in non-resisting media are deflected from rectilinear motion into curved orbits, are to one another as those sagittæ of arcs described in equal times, which converge to the centre of force and bisect the chords, when those arcs are indefinitely diminished.

For the diagonals of the parallelograms $ABCV$, $DEFZ$ bisect each other, and these sagittæ are halves of the diagonals BV, EZ when the arcs are indefinitely diminished. [And the same will be true whether ABC and DEF be parts of the same or of different

orbits described by bodies of equal mass, if the arcs be described in equal times].

Cor. 5. And therefore the accelerating effects of the same forces are to that of the force of gravity as those sagittæ are to vertical sagittæ of the parabolic arcs which projectiles describe in the same time.

Cor. 6. All the same conclusions are true by the Second Law of Motion, when the planes, in which the bodies move together with the centres of force which are situated in those planes, are not at rest, but are moving uniformly and parallel to themselves.

The statement of the proposition in the original Latin is,

" Areas, quas corpora in gyros acta radiis ad immobile centrum virium ductis describunt, et in planis immobilibus consistere, et esse temporibus proportionales."

Observations on the Proposition.

142. In all cases of motion of bodies it is of great importance for the student to distinguish between the forces themselves under the action of which the bodies may be moving, and the effects which these forces produce.

It is only by an examination of the motion of a body that we are able to infer that it is, or is not, acted on by any force; if we find that the body is moving with uniform velocity in a straight line, we infer that it is, during such motion, acted upon by no force, or that the forces which are acting upon it are in equilibrium; if we find that there is any change of direction or velocity, gradual or abrupt, we infer that the body is moving under the action of some force or forces; if the change be gradual, we infer that such forces are *finite*, by which we mean that the forces require a finite time to produce a finite change whether of direction or velocity; if, on the contrary, the change be abrupt, we infer that the forces are what are called *impulsive*, that is, such as produce a finite change in an instant.

Since then, in order to make any inference with respect to the forces supposed to act, a clear conception of the motion of

a body must be first attained, it becomes necessary for the student to be able to describe the motion of a particle of matter as he would that of a point, independently of the causes of such motion.

In doing this he must give a geometrical description of the line traced by the point either in a plane or in space, and then he must describe the rate, uniform or variable, with which this line is traversed.

He may then proceed to attribute any change of direction or velocity to the action of forces upon the particle whose motion he has been examining.

143. In accordance with this method of separating the geometry of the motion from the causes of the deviations, the first proposition would be stated in such a manner as the following:

" When a point moves in a curve, in such a manner that the accelerations at every point are in the direction of a fixed point, the areas, which it describes by radii drawn to the fixed point to which the accelerations tend, are in one fixed plane, and are proportional to the times of describing them."

And, generally, if the words *force* and *body*, employed by Newton, be replaced by *acceleration* and *point*, the resulting statements will be in accordance with this geometrical method of description. It will then be easy to use such terms in the proofs as will not imply, in the manner of expression, the action of force; thus, instead of saying " let a centripetal force tending to S act upon the body by a single instantaneous impulse," we may use the words " let a finite velocity be communicated to the point in the direction of S."

144. It should be carefully observed that, before proceeding to the limit, it is proved that *any* polygonal areas $SADS$, $SAFS$, are proportional to the times of description of their perimeters; so that ultimately these areas become *finite* curvilinear areas, described in *finite* times.

145. In proceeding to the ultimate state of the hypothesis, it is concluded readily from Lemmas II. and III. that the curvilinear areas are the limits of the polygons; but a greater

difficulty arises in the transition from the *discontinuous* motion under the action of instantaneous impulsive forces to the *continuous* motion under the action of a continuous force tending to *S*. For, in the curvilinear path of the body which is the limit of the perimeter of the polygon, the direction of the motion at the angular points of the polygon is different, and also the deflection from the direction of motion is twice as great in the polygon as it is in the curve.

Now, although we may assume that the curvilinear limit of the perimeter of the polygon may be described under the action of some force, is that force the same which is the limit of the series of impulses?

The centripetal force supposed to act with a simple instantaneous impulse, "impulsu unico et magno," is supposed to generate a finite velocity at once, which effect a finite force cannot produce.

If, instead of this imaginary impulse, we suppose a force finite, but very great, and acting for a very short time, the effect upon the figure would be to round off the angular points of the polygon.

The transition from the impulses to the continuous force, in the ultimate form of the hypothesis, must be considered as axiomatic, like the ultimate equality of the ratio of the finite arc to the perimeter of the inscribed polygon.

146. We can, however, shew that if the curvilinear limit of the polygon be described under the action of some continuous force tending to *S*, the effect of this force, estimated by the quantity of motion generated in the interval between the impulses, will be ultimately the same as that generated by the impulse.

Consider first the geometrical properties of the limit of the polygonal perimeter. Let BT, CU be tangents at B, C to the curvilinear limit, and let Cc intersect BT in T, fig. page 136.

Now, since Cc ultimately vanishes compared with Bc, BC and Bc or AB and BC are ultimately in a ratio of equality, and Cc is ultimately bisected by BT, by (2) page 102; also, $CU = BU = UT$ ultimately, by (1) page 102.

Consider next the effects produced by the different kinds of force which act in the two cases.

In the polygonal path, the impulsive force at B generates a velocity with which the body describes Cc in the time t in which AB or BC is described, the measure of the effect of the impulse is therefore the velocity $\dfrac{Cc}{t}$.

In the curvilinear path, the deflection from the direction BT at B, in the same time t, is TC, by means of the continuous action of finite forces, and if we suppose the force ultimately uniform in magnitude and direction, the measure of the accelerating effect of the force will be $\dfrac{2TC}{t^2}$, and the velocity generated in that time will be $\dfrac{2TC}{t^2}.t=\dfrac{Cc}{t}$.

Hence the effects of the finite and impulsive forces, measured by the quantity of motion produced, are the same.

147. We can also shew that a continuous force, which generates the same quantity of motion as the impulse at B in the time from B to C, would cause the body on arriving at C to move in the direction of the tangent to the curvilinear limit of the perimeter.

For the velocity due to the action of the finite force at the end of time t being ultimately $\dfrac{2TC}{t}$ in the direction TC, and that in the direction BT being $\dfrac{BT}{t}=\dfrac{2TU}{t}$; therefore TC, UT represent the velocities in those directions; therefore UC is the direction of motion at C, that is, the body moves in the direction of the tangent at C.

148. Cor. 1. The corollary may be proved directly from the proposition, for the proportionality of the areas to the times of describing them will be true if the force suddenly cease to act, in which case the body will proceed in the direction of the tangent.

Let V be the velocity at the point A, ASB the curvilinear area described in any time T, $AT=V.T$ the space described if the force cease to act. Join ST and draw SY perpendicular

to AT, then area $ASB=$ triangle $SAT=\frac{1}{2}V.T\times SY$, also area $ASB \propto T$; therefore V varies inversely as SY.

Again, if h be twice the area described in the unit of time
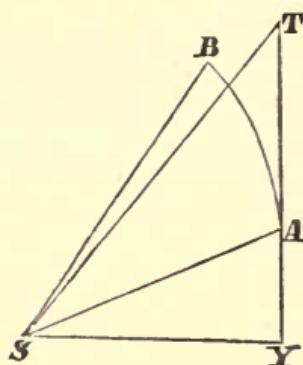
employed in estimating the accelerating effect of the force tending to S and the velocity of the body,

$$2 . \text{area} \, SAB = hT; \ \therefore \ h = V.SY.$$

By the use of this area the proportions employed in subsequent propositions by Newton may be converted into equations, for the convenience of calculation.

If bodies move in curves for which the areas, described in the same time, are not equal, $V \propto \dfrac{h}{SY}$.

149. COR. 4. The statement in this corollary requires modification, for, unless the forces be considered only with reference to their accelerating effects, or unless the bodies be supposed of equal mass, the forces will not be proportional to the sagittæ.

150. COR. 5. The object of this corollary is to determine the numerical measure of the central force which governs the motion of a body, when the circumstances of the motion are known; for it supplies us with the ratio of this force to the force of gravity on the same body at any place, the measure of which can be determined by experiment.

Applications of the Proposition.

151. PROP. *When the force, instead of tending to a fixed point, acts in parallel lines, the property of the motion enunciated*

in teh proposition may be replaced by the property that the resolved part of the space described perpendicular to the direction of the force is proportional to the times.

This is immediately deducible from the second law of motion, since there is no force in the direction perpendicular to that of he forces, and the velocity in that direction is uniform.

That this is the result of the properties in the proposition may be shewn by removing the centre of force to an infinite distance.
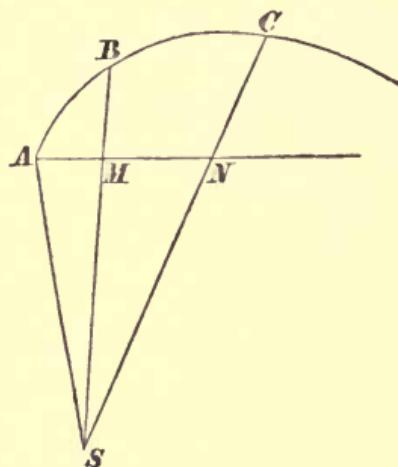
Let S be the centre of force, AMN perpendicular to SB, the area $ABCS$ is proportional to the time of describing AC, and the areas $AMNS$ and $ABCS$ are ultimately equal when S is removed to an infinite distance in BMS, hence the triangle ASN is proportional to the time, and therefore the base AN, which varies as the triangle ASN, is also proportional to the time, and therefore, since CN is ultimately perpendicular to AN, the proposition is proved.

152. PROP. *If a body describe a curvilinear orbit about a force tending constantly to a fixed point, the area described in a given time will be unaltered, if the force be suddenly increased ar diminished, of if the body be acted on at any moment by an impulsive force tending to that point.*

For, if in the polygon the impulse at any point B be increased or diminished by any force tending to or from S, the only effect will be to remove the vertex C of the triangle SBC to

some other point in the line cC parallel to BS, hence the area will be unaltered, and the argument which establishes the equality of polygonal areas in a given time will proceed as before. Hence in the limit the curvilinear areas described in a given time will be unaltered.

If the new force introduced at B be impulsive, the angle ABC will remain less than two right angles when we proceed to the limit, and the two parts of the curve will cut one another at a finite angle.

Hence, in any calculation made upon supposition of such changes of force, the value of h, Art. 148, will be the same before and after the change of the force.

Apses.

153. DEF. In any orbit described under the action of a force tending to a fixed centre, a point at which the direction of the motion is perpendicular to the central distance is called an *apse*, the distance from the centre is called an *apsidal distance*, and the angle between consecutive apsidal distances is called an *apsidal angle*.

Thus, in the ellipse about the centre, the four extremities of the axes are apses; there are two different apsidal distances, and every apsidal angle is a right angle.

In the ellipse about a focus, the apses are at the greatest and least distances, and the apsidal angle is two right angles.

154. *In a central orbit described under the action of forces tending to a fixed point, each apsidal distance will divide the orbit symmetrically, if the forces be always equal at equal distances.*

It is easily shewn that, in any orbit described by a body under the action of forces tending to a fixed point, the forces depending only upon the distance, if a second body be projected at any point with the velocity of the first in the opposite direction, it will proceed to describe the same orbit in the reverse direction, under the action of the same forces.

For, let ABC be a portion of the polygon whose limit is the curvilinear path of the body, and produce AB to c, and CB to a, making $Bc = AB$, and $Ba = CB$.

The impulse at B is measured by cC when the body describes ABC, and if the motion be reversed, the same impulse at B would cause the body to move in BA, with the velocity which it had in AB, since $aA = cC$. And the same is true
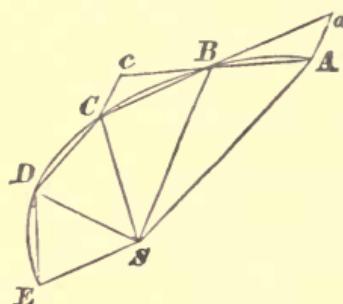
throughout the polygonal path, hence the assertion is true for the whole path, described under the action of impulses which are always the same at the same points, and therefore, proceeding to the limit, the statement made for any orbit is proved.

Hence, since the forces are equal at equal distances on both sides of the apse, the path of the body from an apse being similar and equal to the path which would be described if the velocity were reversed at the apse, is similar to the path described in approaching the apse; whence the proposition is established.

155. *There are only two different apsidal distances, and all apsidal angles are equal.*

For, after passing a second apse, the curve being symmetrical on both sides, a third apse will be in such a position that the apsidal distance is the same as for the first apse, and all the apsidal angles are shewn similarly to be equal.

156. COR. Hence a central orbit can never re-enter itself unless the ratio of the apsidal angle to a right angle be commensurable, and if it be so, the curve will always re-enter.

Illustrations.

(1) *If a body describe an ellipse under the action of a force tending to one of the foci, the square of the velocity varies inversely*

as the distance from that focus, and directly as the distance from the other.

$$\text{For } BC^2 : SY^2 :: HZ : SY :: HP : SP;$$

$$\therefore (\text{vel.})^2 \propto \frac{1}{SY^2} \propto \frac{HP}{SP}.$$

(2) *The velocity is greatest when the body is at the extremity of the major axis which is nearer to the focus to which the force tends, and least at the other extremity.*

For SY is the least in the first and greatest in the second position.

(3) *The velocity at an extremity of the minor axis is a geometric mean between the greatest and least velocities.*

For at this point $HZ = BC$, and at the extremities of the major axis the values of HZ are Sa and SA, and $BC^2 = SA . Sa$.

(4) *In the equiangular spiral described under the action of a force tending to the focus, the velocity* $\propto \dfrac{1}{SP}$.

$$\text{For, } SY \propto SP.$$

(5) *If the force tend to the centre of the elliptic orbit described by a body, the time between the extremities of conjugate diameters will be constant.*

$$\text{For the area } PCD \text{ is constant.}$$

(6) *The velocity at any point of an ellipse about a force tending to a focus is compounded of two uniform velocities, one perpendicular to the radius vector, and the other perpendicular to the major axis.*

Let S be the centre of force, SY, HZ perpendiculars on the tangent at P, join SP, CZ. Then HZ, ZC parallel to PS, and
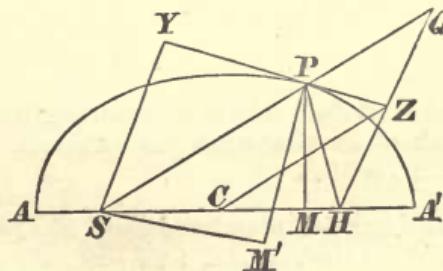
CH are perpendicular to the three directions; therefore the velocity represented by HZ in magnitude is the resultant of the two represented by CZ and HC; but the velocity perpendicular to $HZ = \dfrac{h}{SY} = \dfrac{h}{b^2} . HZ$; therefore the velocities perpendicular to HC and CZ are $\dfrac{h}{b^2} ae$ and $\dfrac{h}{b^2} a$.

XVIII.

1. If different bodies be projected with the same velocity from a given point, all being attracted by forces tending to one fixed point, shew that the areas described by the lines drawn from the fixed point to the bodies will be proportional to the sines of the angles of projection.

2. When a body describes a curvilinear orbit under the action of a force tending to a fixed point, will the direction of motion or the curvature of the orbit at any point be changed, if the force at the point receive a finite change?

3. A body moves in a parabola about a centre of force in the vertex, shew that the time of moving from any point to the vertex varies as the cube of the distance of the point from the axis of the parabola.

4. In a parabolic orbit described round a force tending to the focus, shew that the velocity varies inversely as the normal at any point. Shew also that the sum of the squares of the velocities ta the extremities of a focal chord is constant.

5. If the velocity at any point of an ellipse described about the centre can be equal to the difference of the greatest and least velocities, the major axis cannot be less than double of the minor.

6. If an ellipse be described under the action of a force tending to the centre, shew that the velocity will vary directly as the diameter conjugate to that which passes through the body; also that the sum of the squares of the velocities at the extremities of conjugate diameters will be constant.

7. In an ellipse described round a force tending to the focus, compare the intervals of time between the extremities of the same latus rectum, when $AC = 2CS$.

8. In the ellipse described about the focus S, $ASHA'$ being the major axis, time in AB : time in BA' :: $\pi - 2e$: $\pi + 2e$.

9. If the velocities at three points in an ellipse described by a particle, the acceleration of which tends to either of the foci, be in arithmetical progression, prove that the velocities at the opposite extremities of the diameters passing through these points will be in harmonical progression.

10. If v_1, v_2 be the velocities at the extremities of a diameter of an ellipse described about the focus, and u the velocity at either of those points when it is described about the centre, prove that $u\,(v_1 + v_2)$ will be constant.

11. In a central orbit, the velocity of the foot of the perpendicular from the centre of force on the tangent varies inversely as the length of the chord of curvature through the centre of force.

12. A particle is describing a parabola about its focus S; if P and Q be two points of its path, shew that its velocity at Q will be compounded of the velocity at P and a velocity which will be constant if the angle PSQ be constant.

XIX.

1. A body describes a parabola about a centre of force in the focus; shew that its velocity at any point may be resolved into two equal constant velocities, respectively perpendicular to the axis and to the focal distance of the point.

2. A body describes an ellipse under the action of a central force tending to one of the foci; shew that the sum of the velocities at the extremities of any chord parallel to the major axis varies inversely as the diameter parallel to the direction of motion at those points.

3. A body moves in an ellipse under the action of a force tending to the centre; shew that the component of the velocity at any point perpendicular to either focal distance is constant; and that the sum of the squares of the velocities, at the extremities of any pair of semi-conjugate diameters, resolved in any given direction is constant.

4. In an ellipse described about a focus, the time of moving from the greatest focal distance to the extremity of the minor axis is m times that from the extremity of the minor axis to the least focal distance; find the eccentricity, and shew that, if there be a small error in m, the corresponding error in the eccentricity will vary inversely as $(1 + m)^2$.

5. If the velocity of a body in a given elliptic orbit be the same at a certain point, whether it describe the orbit in a time t about

one focus, or in a time t' about the other, prove that, $2a$ being the major axis, the focal distances will be $\dfrac{2at'}{t+t'}$ and $\dfrac{2at}{t+t'}$.

6. A body describes a parabola about the focus; if the segments PS, Sp of the focal chord PSp be in the ratio $n : 1$, prove that the time in pA : time in AP :: $3n + 1 : n^2(n+3)$.

7. If SY be perpendicular to the tangent to a curve at P, and P and Y both move as if under the action of a central force tending to S, prove that the radius of curvature at P will vary as SY.

8. If P, Q be any two points in an ellipse described by a particle under the action of a force tending to the centre, prove that the velocity acquired in passing from P to Q will be in the direction QP', where P' is the other extremity of the diameter through P.

9. Two points P, P' are moving in the same ellipse, in the same directions, with accelerations tending to the centre C; shew that the relative velocity of one with regard to the other is parallel and proportional to CT, where T is the point of intersection of the tangents at P and P'. If the points move in opposite directions, what will be their relative velocity?

10. Two particles revolve in the same direction in an oval orbit round a centre of force S, which divides the axis unequally, starting simultaneously from the extremities of a chord PQ, drawn through S. Prove that, when they first arrive in positions R, T respectively, such that the angle RST is a minimum, the time from R to the next apse will be an arithmetic mean between the times from P to the next apse and to Q from the last apse.

11. Two equal particles are attached to the extremities of a string of length $2l$, and lie in a smooth horizontal plane with the string stretched; if the middle point of the string be drawn with uniform velocity v in a direction perpendicular to the initial direction of the string, shew that the path of each particle will be a cycloid, and that the particles will meet after a time $\dfrac{l\pi}{2v}$.

12. If the velocity in a central orbit can be resolved into two constant components, one perpendicular to the radius vector, and the other to a fixed straight line, shew that the curve must be a conic.

13. The velocity in a cardioid described about a force tending to the pole varies in the inverse sesquiplicate ratio of the distance.

14. The velocity in the lemniscate varies inversely as the cube of the central distance, when a particle moves in the curve round a force tending to the centre.

PROP. II.   THEOREM II.

Every body, which moves in any curve line described in a plane, and describes areas proportional to the times of describing them about a point either fixed or moving uniformly in a straight line, by radii drawn to that point, is acted on by a centripetal force tending to the same point.

Case 1. Let the time be divided into equal intervals, and in the first interval let the body describe AB with uniform velocity, being acted on by no force; in the second interval it would, if no force acted, proceed to c in AB produced, describing Bc equal to AB; and the triangles ASB, BSc would be equal.   But
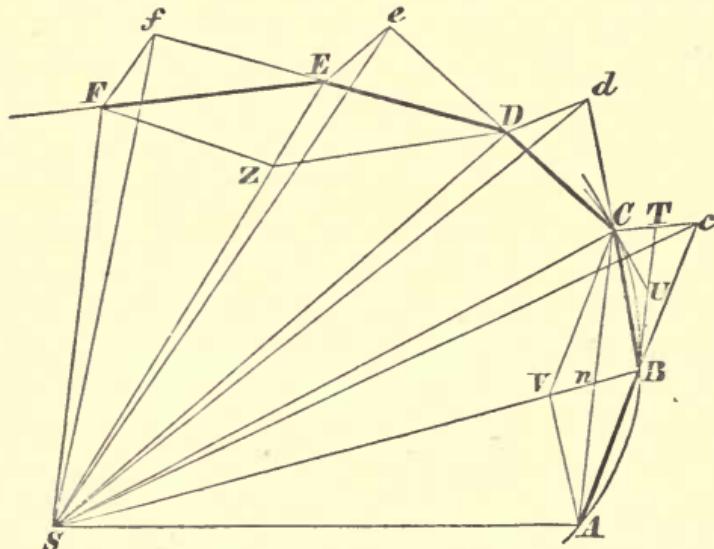
when the body arrives at B, let a force, acting upon it by a single impulse, cause the body to describe BC in the second interval of time, so that the triangle BSC is equal to the triangle ASB, and therefore also to the triangle BSc; therefore BSC and BSc are between the same parallels, hence BS is

parallel to cC, and therefore BS was the direction of the impulse at B.

Similarly, if at C, D, ... the body be acted on by impulses causing it to move in the sides CD, DE, ... of a polygon, in the successive intervals, making the triangles CSD, DSE, ... equal to ASB and BSC, the impulses can be shewn to have been in the directions CS, DS, ... Hence, if *any* polygonal areas be described proportional to the times of describing them, the impulses at the angular points will all tend to S.

The same will be true if the number of intervals be increased and their length diminished indefinitely, in which case the series of impulses will approximate to a continuous force tending to S, and the polygons to curvilinear areas, as their limits. Hence the proposition is true for a *fixed* centre.

Case 2. The proposition will also be true if S be a point which moves uniformly in a straight line, for, by the second law of motion, the relative motion will be the same, whether we suppose the plane to be at rest, or that it moves together with the body which revolves and the point S, uniformly in one direction.

Cor. 1. In non-resisting media, if the areas be not proportional to the times, the forces will not tend to the point to which the radii are drawn, but will deviate *in consequentiâ*, *i.e.* in that direction towards which the motion takes place, if the description of areas be accelerated; but if it be retarded, the deviation will be *in antecedentiâ*.

Cor. 2. And also in resisting media, if the description of areas be accelerated, the directions of the forces will deviate from the point to which the radii are drawn in that direction towards which the motion takes place.

SCHOLIUM.

A body may be acted on by a centripetal force compounded of several forces. In this case, the meaning

of the proposition is, that that force, which is the resultant of all, tends to S. Moreover, if any force act continually in a line perpendicular to the plane of the areas described, this force will cause the body to deviate from the plane of its motion, but will neither increase nor diminish the amount of area described, and therefore must be neglected in the composition of the forces.

Observations on the Proposition.

157. The description of an area round a point in motion may be explained by the following construction for the relative orbit, in the case of motion about a point which is itself moving uniformly in a straight line.

Let SS' be the line in which S moves uniformly, and let the body move from A to B in the same time that S moves from S to S', and let P, σ be simultaneous positions of the body and of S.
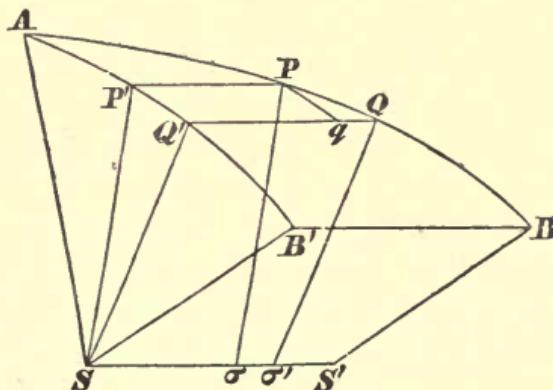
If PP' be drawn equal and parallel to σS, and the same construction be made for every point in the path of the body, the curve $AP'B'$, which is the locus of P', will be the orbit which the body would appear to describe to an observer at S, who referred all the motion to the body; for SP' will be equal and parallel to σP, and therefore the distance of the body, and the direction in which it is seen, will be the same in the two cases.

If Q, Q' be corresponding points near P and P', and the force at σ be supposed to act impulsively, the relative motion round σ will be unaltered if we apply to both P and σ velocities equal to

X

that of σ and in a contrary direction, but in this case σ will be reduced to rest and the velocity of P will be the velocity relative to σ. Take PQ and $\sigma\sigma'$, which are described in the same time, to represent the velocities of P and σ, and let Qq be equal and parallel to $\sigma'\sigma$, then Pq represents the velocity of P relative to σ; and, since $Q'q = S\sigma - \sigma'\sigma = P'P$, $P'Q'$ is equal and parallel to Pq, and therefore the velocity in the orbit AB' about S at rest is equal to the relative velocity about S in motion.

158. Cor. 1. Reverting to the polygonal area, if the tri-

angle SBC' be greater than the triangle SAB, the impulse at B will not be in the direction BS, but BU, parallel to cC', that is, if the areas be not proportional to the times but be in an increasing ratio, the direction of the force will deviate towards the direction in which the description of areas is accelerated; and *vice versâ*, when the description is retarded.
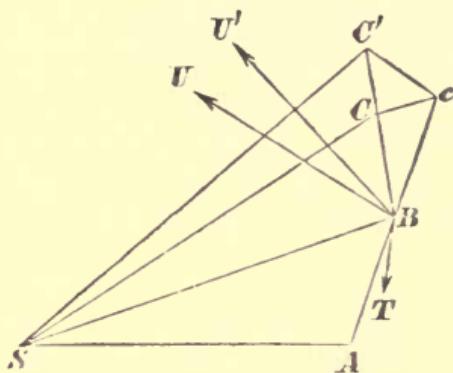
159. Cor. 2. The effect of a resisting medium is to retard the motion, or, supposing it the limit of a series of impulses, we must conceive an impulse at B, in the case of the polygon, in the direction BA; if therefore the description of areas be accelerated, the impulse applied at B in the direction BU' must act still further *in consequentiâ* than that in BU, in order that, with the impulse corresponding to the resistance of the medium, it may produce a resultant impulse in the direction of BU. The effect of the resistance alone is to retard the description of areas.

If the force act *in consequentiâ*, the resultant of this force

and the resistance of the medium may act in the direction BS, and the proportionality of the areas to the times be preserved.

160. PROP. *Let ABCDE be any plane curve, S any point in the plane, to shew that, generally, the curve can be described under the action of a force tending to or from S, with finite velocity, the velocity at any given point being any given velocity.*

For arcs AB, BC, ... can be measured from any point A, along the curve, such that the areas SAB, SBC,... are all equal,
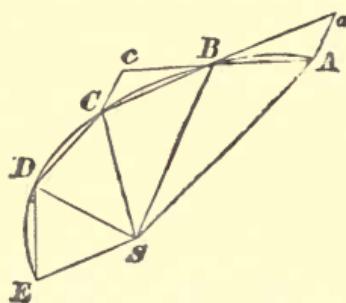
and of any magnitude. Also a body can be made, by some force to move along the curve with finite velocity, so as to describe the arcs AB, BC, ... in equal times, unless the tangent to one of the arcs, as DE, pass through S, in which case, if the arcs be indefinitely diminished, DE, AB will not be finite ultimately.

Hence by Prop. II. a body *can* move with finite velocity under the action of some force tending to or from S, *generally*.

161. NOTE 1. Since in making the motion of the body such that it shall describe equal areas in equal times we are only concerned with the ratio of the velocities, the velocity at any point A may be any given velocity.

162. NOTE 2. Or if we please we may suppose the force at any point any given force ; for, in the case of the polygon, the velocity generated by the impulse at B is to the velocity in AB as cC to Bc, hence the impulse at B may be of any magnitude if we choose the velocity in AB properly.

163. NOTE 3. The ratio of the velocities will be the same at two given points, for all forces tending to a given centre, under the action of which the curve can be described.

164. NOTE 4. Hence a body can move throughout any ellipse under the action of a centripetal force tending to the centre or focus, the force depending only on the distance, since in these cases the curve is symmetrical on opposite sides of any apse; or about any point within the ellipse, if the forces do not depend only on the distance, since no point within an ellipse lies on any tangent.

165. NOTE 5. In the case of an oval, S being an external point, a body can move with finite velocity under the action of a force tending to the point S, in the portion which is concave to S, and from S, in that which is convex to S, but not from one portion to the other.

XX.

1. If an ellipse be described so that the sum of the areas swept out by radii drawn to the vertices is proportional to the times of describing them, prove that the resultant acceleration will tend to the centre.

2. A body is moving in a parabola, and the time from the vertex to any point varies as the cube of the ordinate; shew that this motion could be caused by the action of a central force.

3. A body was moving in a circle, and it was observed that the time of describing any arc from a fixed point varied as the sum of the arc and the perpendicular distance from one extremity of the arc on the diameter through the other; shew that the body was acted on by a central force.

4. A heavy particle falls from the cusp to the vertex of a cycloid, whose axis is vertical; shew that a particle could describe the cycloid in the same manner under the action of a constant force directed to a certain moving point.

5. From the centre of a planet a perpendicular is let fall upon the plane of the ecliptic; prove that the foot of this perpendicular will move as if it were a particle acted on by a force tending to the sun's centre.

PROP. III. THEOREM III.

Every body, which describes areas proportional to the times of describing them by radii drawn to the centre of another body which is moving in any manner whatever, is acted on by a force compounded of a centripetal force tending to that other body, and of the whole accelerating force which acts upon that other body.

Let the first body be L, the second T, T moves under the action of some force P, L under the action of
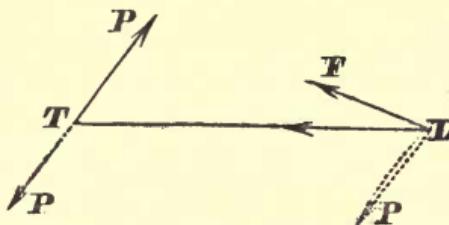
another force F. At every instant apply to both bodies the force P in the contrary direction to that in which it acts, as represented by the dotted arrows.

L will continue to describe about T, as before, areas proportional to the times of describing them, and since there is now no force acting on T, T is at rest or moves uniformly in a straight line.

Therefore, by Theorem II., the resultant of the force F and the force P applied to L tends to T.

Hence F is compounded of a centripetal force tending to T, and of a force equal to that which acts on T. Q.E.D.

Cor. 1. Hence, if a body L describe areas proportional to the times of describing them by radii drawn to another body T; and from the whole force which acts upon L, whether a single force or compounded of several forces, be taken away the whole accelerating force which acts upon the other body T; the whole

remaining force, which acts upon L, will tend to the other body T as a centre.

COR. 2. And if these areas be very nearly proportional to the times of describing them, the remaining force will tend to the other body very nearly.

COR. 3. And, conversely, if the remaining force tend very nearly to the other body T, the areas will be very nearly proportional to the times.

COR. 4. If the body L describe areas which are very far from being proportional to the times of describing them, by radii drawn to another body T, and that other body T be at rest, or move uniformly in a straight line, then either there will be no centripetal force tending to that other body T, or such centripetal force will be compounded with the action of other very powerful forces, and the whole force compounded of all the forces, if there be many, may be directed towards some other centre fixed or moving.

The same will hold, when the other body moves in any manner whatever, if the centripetal force spoken of be understood to be that which remains after taking away the whole force acting upon the other body T.

SCHOLIUM.

Since the equable description of areas is a guide to the centre to which that force tends, by which a body is principally acted on, and by which it is deflected from rectilinear motion, and retained in its orbit, we may, in what follows, employ the equable description of areas as a guide to the centre, about which all curvilinear motion in free space takes place.

Illustration.

166. As an illustration of the last propositions and their corollaries, we may state some of the observed facts in the motion of the Moon, Earth, and Sun, and make the deductions corresponding to them.

Suppose the Moon's orbit relative to the Earth to be nearly circular, and let $ABCD$ be this orbit, E the Earth.
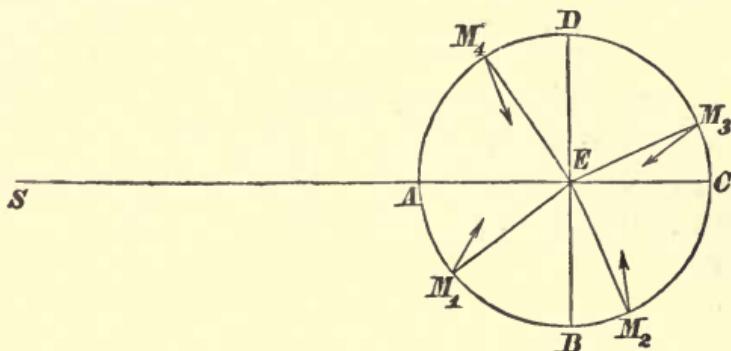
(1) The areas described by the radii drawn from the Moon to the Earth are nearly proportional to the times of describing; hence the resultant force on the Moon tends nearly to E.

(2) If ES the line joining the centres of the Earth and Sun meet the Moon's relative orbit about the Earth in A, C, and DEB be perpendicular to DS, the description of areas will be accelerated as the Moon moves from D to A and from B to C, and retarded from A to B and from C to D; hence the direction of the resultant force on the Moon in the positions M_1, M_2, M_3, M_4, will be in the directions of the arrows slightly inclined to the radii drawn to E.

From these observed facts, we see that when the force, under the action of which E moves, is applied to the Moon in the contrary direction, the remaining force tends in the directions of the arrows.

By the supposition that the Earth and Moon are acted on by forces tending to the sun, whose distance compared with EM is very great, and that the differences of the forces on these bodies are not very great, the accelerating effect of the force on the Moon in DAB being greater than that on the Earth, and in BCD less, the circumstances of the description of areas in the motion of the Moon are accounted for.

PROP. IV. THEOREM IV.

The centripetal forces on equal bodies, which describe dif-
ferent circles with uniform velocity, tend to the centres of
the circles, and are to each other as the squares of arcs
described in the same time, divided by the radii of the
circles.

The bodies move uniformly, therefore the arcs described
are proportional to the times of describing them; and
the sectors of circles are proportional to the arcs on
which they stand, therefore the areas described by
radii drawn to the centres are proportional to the
times of describing them; hence, by Prop. II., the
forces tend to the centres of the circles.

Again, let AB, ab be small arcs described in equal times,
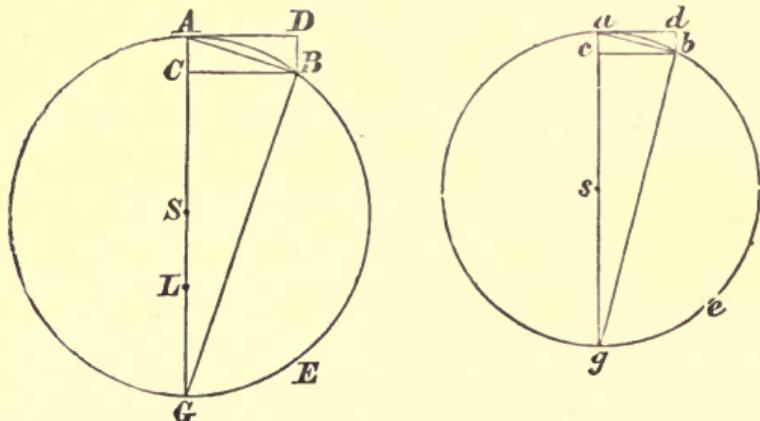
AD, ad tangents at A, a; $ACSG$, $acsg$ diameters
through A, a. Join AB, ab, and draw BC, bc per-
pendicular to AG, ag.

When the arcs AB, ab are indefinitely diminished, since
AC, ac are sagittæ of the double of arcs AB, ab
described in equal times, they are ultimately, by
Prop. I., Cor. 4, as the forces at A and a.

But $AC . AG = (\text{chd } AB)^2$ and $ac . ag = (\text{chd } ab)^2$;

∴ force at A : force at a :: AC : ac ultimately,

$$:: \frac{(\text{chd }AB)^2}{AG} : \frac{(\text{chd }ab)^2}{ag} :: \frac{(\text{arc }AB)^2}{AG} : \frac{(\text{arc }ab)^2}{ag}, \text{ Lem. VII.}$$

Take AE, ae two arcs described in *any* equal finite times, then $AE : ae :: AB : ab$, since the bodies move uniformly, and this is also true in the limit;

$$\therefore \text{ force at } A : \text{force at } a :: \frac{AE^2}{AS} : \frac{ae^2}{as}. \quad \text{Q.E.D.}$$

COR. 1. Since these arcs are proportional to the velocities of the bodies, the centripetal forces will be in the ratio compounded of the duplicate ratio of the velocities directly, and the simple ratio of the radii inversely.

That is, if V, v be the velocities in the two circles, R, r the radii, F, f the centripetal forces, $AE : ae :: V : v$;

$$\therefore F : f :: \frac{V^2}{R} : \frac{v^2}{r}.$$

COR. 2. And since the circumferences of the circles are described in their periodic times, the velocities are in the ratio compounded of the ratio of the radii directly and the ratio of the periodic times inversely; hence the centripetal forces are in the ratio compounded of the ratio of the radii directly, and of the ratio of the squares of the periodic times inversely.

If P, p be the periodic times in the two circles respectively,

$$V : v :: \frac{2\pi R}{P} : \frac{2\pi r}{p} :: \frac{R}{P} : \frac{r}{p};$$

$$\therefore F : f :: \frac{V^2}{R} : \frac{v^2}{r} :: \frac{R}{P^2} : \frac{r}{p^2}.$$

COR. 3. Hence, if the periodic times be equal, and therefore the velocities proportional to the radii, the centripetal forces will be as the radii; and conversely.

If $P = p$, then $V : v :: R : r$;

$$\therefore\ F : f :: \frac{V^2}{R} : \frac{v^2}{r} :: R : r.$$

Cor. 4. Also if the periodic times be in the subduplicate ratio of the radii, the centripetal forces will be equal. That is, if $P^2 : p^2 :: R : r$, then $F = f$, by Cor. 2.

Cor. 5. If the periodic times be as the radii, and therefore the velocities equal, the centripetal forces will be reciprocally as the radii; and conversely.

Cor. 6. If the periodic times be in the sesquiplicate ratio of the radii, and therefore the velocities reciprocally in the subduplicate ratio of the radii, the centripetal forces will be reciprocally as the squares of the radii; and conversely.
That is, if $P^2 : p^2 :: R^3 : r^3$,

$$\text{then}\ V^2 : v^2 :: \frac{R^2}{P^2} : \frac{r^2}{p^2} :: \frac{1}{R} : \frac{1}{r};$$

$$\therefore\ F : f :: \frac{V^2}{R} : \frac{v^2}{r} :: \frac{1}{R^2} : \frac{1}{r^2}.$$

Cor. 7. And, generally, if the periodic times vary as any power R^n of the radius R, and, therefore, the velocity vary inversely as the power R^{n-1}, the centripetal force will vary inversely as R^{2n-1}; and conversely.

Cor. 8. All the same proportions can be proved concerning the times, velocities, and forces, by which bodies describe similar parts of any figures whatever, which are similar and have centres of force similarly situated, if the demonstrations be applied to those cases, *uniform description of areas* being substituted for *uniform velocity*, and *distances of the bodies from the centres of force* for *radii of the circles*.

Let AE, ae be similar arcs of similar curves described by bodies about forces tending to similarly situated points S, s; and let AB, ab be small arcs described

in equal times; BD, bd subtenses parallel to SA, sa; AV, av chords of curvature at A, a, so that

$$AV : av :: AS : as.$$
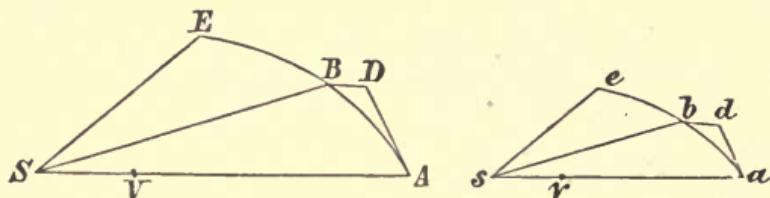
Then, force at A : force at a :: DB : db, ultimately,

$$:: \frac{AB^2}{AV} : \frac{ab^2}{av} :: \frac{AB^2}{SA} : \frac{ab^2}{sa}, \text{ ultimately};$$

and if V, v be the velocities at A, a since AB, ab are described in equal times, $AB : ab :: V : v$, ultimately;

$$\therefore \text{ force at } A : \text{ force at } a :: \frac{V^2}{SA} : \frac{v^2}{sa}, \text{ as Cor. 1.}$$

Again, if AB, ab be small similar arcs described in times T, t, instead of being arcs described in equal times, and P, p be the times of describing similar finite arcs AE, ae,

$$T : P :: \text{area} ASB : \text{area} ASE :: \text{area} asb : \text{area} ase :: t : p;$$

therefore, when AB, ab are indefinitely diminished, $T : t :: P : p$.

Hence, $F : f :: \dfrac{V^2}{SA} : \dfrac{v^2}{sa} :: \dfrac{AB^2}{T^2 . SA} : \dfrac{ab^2}{t^2 . sa}$ ultimately,

$$:: \frac{SA}{T^2} : \frac{sa}{t^2} :: \frac{SA}{P^2} : \frac{sa}{p^2}, \text{ as Cor. 2.}$$

Cor. 9. It follows also from the same proposition, that the arc which a body, moving with uniform velocity in a circle under the action of a given centripetal force, describes in any time, is a mean proportional between the diameter of the circle, and the space through which the body would fall from rest under the action of the same force and in the same time.

For, let AL be the space described from rest in the same time as the arc AE, then since, if BD be perpendicular to the tangent at A, BD will be ultimately the space described by the body, under the action of the force at A, in the time in which the body describes the arc AB, and the times are proportional to the arcs;

$$\therefore AL : BD :: AE^2 : AB^2 ;$$

$$\therefore AL . AG : BD . AG :: AE^2 : AB^2 ;$$

and $BD . AG = (\text{chd} AB)^2 = (\text{arc} AB)^2$, ultimately;

therefore $AL . AG = AE^2$, or $AL : AE :: AE : AG$.

Q. E. D.

SCHOLIUM.

The case of the sixth Corollary holds for the heavenly bodies, and on that account the motion of bodies acted upon by a centripetal force, which decreases in the duplicate ratio of the distance from the centre of force, is treated of more fully in the following section.

Moreover, by the aid of the preceding proposition and its corollaries, the proportion of a centripetal force to any known force, such as gravity, can be obtained. For, if a body revolve in a circle concentric with the earth by the action of its own gravity, this gravity is its centripetal force.

But, from the falling of heavy bodies, by Cor. 9, both the time of one revolution and the arcs described in any given time are determined.

And by propositions of this kind Huygens, in his excellent tract, *De Horologio Oscillatorio*, compared the force of gravity with the centrifugal force of revolving bodies.

The preceding results may be proved in this manner. In any circle let a regular polygon be supposed to be described of any number of sides. And if a body moving with a given velocity along the sides of the polygon be reflected by the circle at each of its angular points, the force with which it impinges on

the circle at each of the reflections will be proportional to the velocity; and therefore the sum of the forces, in a given time, will vary as the velocity and the number of the reflections conjointly. But if the number of sides of the polygon be given, the velocity will vary as the space described in a given time, and the number of reflections in a given time will vary, in different circles, inversely as the radii of the circles, and, in the same circle, directly as the velocity. Hence, the sum of the forces exerted in a given time varies as the space described in that time increased or diminished in the ratio of that space to the radius of the circle; that is, as the square of that space divided by the radius, and therefore, if the number of sides be diminished indefinitely so that the polygon coincides with the circle, the sum of the forces varies as the square of the arc described in the given time divided by the radius.

This is the *centrifugal force* by which the body presses against the circle, and to this the opposite force is equal, by which the circle continually repels the body towards the centre.

Symbolical representation of Areas, Lines, &c.

167. In the statement of the proposition the words "arcuum quadrata applicata ad radios," in the text of Newton, is rendered the squares of arcs divided by the radii. Such expressions as $\dfrac{AB^2}{AG}$ may be regarded as representations of lines (*e.g.* this expression denotes AC) whose lengths are determined by such constructions as the following:

To AG apply a rectangle whose area is that of the square on AB, and let AC be the side adjacent to AG; AC is thus obtained by applying the square on AB to AG. The propriety of the symbol $\dfrac{AB^2}{AG}$ employed to represent a line AC, assumed from algebra, is obvious, since the number of units of area in the square on AB and in the rectangle whose sides are AG,

AC are the same; hence, if m, n, r be the number of units of length in these lines, $m^2 = n \times r$ and $r = \dfrac{m^2}{n}$.

168. If symbols of this kind, viz. $\dfrac{AB^2}{AG}$, be used in the same manner as a fraction, we may either treat them numerically, considering AB^2 to represent the number of units of area contained in the square on AB, and AG as the number of units of length in AG, and thus apply the rules of Arithmetical Algebra; or we may look upon AB^2 as the absolute representation of an area, and AG as that of a line, in which case $\dfrac{AB^2}{AG}$ would have no meaning except by interpretation. In this interpretation we are guided by the principles upon which Symbolical Algebra is applied to any science, the laws of operation by symbols being the same in Arithmetical and Symbolical Algebra, and the symbols being interpreted so that these laws are not contradicted. Thus if, in the application to Geometry, the symbol A be supposed to denote an area equal to that of a rectangle whose sides are represented by a and b, the assumption that $A = ab$ or ba will imply that $ab = ba$, hence the laws remain the same as in Arithmetical Algebra, and $\dfrac{A}{a} = b$; so that the interpretation is legitimate, that, if a rectangle be applied to a, whose area is A, $\dfrac{A}{a}$ will denote the other side of the rectangle.

Observations on the Proposition.

169. In the statement of the proposition the word 'equal' has been inserted before 'bodies' in order to make the theorem correct, whether we suppose the centripetal force to be estimated with reference to the momentum or the velocity generated.

It would, perhaps, be better to state the proposition as follows: "The resultant of the forces, under the action of which bodies describe different circles with uniform velocity, are centripetal and tend to the centres of the circles, and their accelerating effect are to each other, &c.," for it is not known, prior to the proof, that the forces are centripetal.

170. Cors. 1 and 9. The first corollary asserts that the centripetal forces on bodies moving in different circles vary as $\frac{V^2}{R}$, but the ninth shews that the accelerating effects of the centripetal forces are in each circle equal to $\frac{V^2}{R}$.

For, if V be the velocity, F the accelerating effect of the force in any circle, T the time of describing any arc, VT will be the length of the arc, $\frac{1}{2}FT^2$ will be the space through which the body would move under the action of the same force continued constant, in the same time in which the arc is described,

$$\therefore \ \tfrac{1}{2}FT^2 : VT :: VT : 2R; \ \therefore \ V^2 = FR.$$

171. Scholium. In uniform circular motion the centripetal force is employed in counteracting the tendency of the body to move in a straight line, which it would do, according to the first law of motion, with the uniform velocity which it has at any point of the circle, if the centripetal force were suddenly to cease to act. This tendency to recede is called a centrifugal force improperly; for the effect of a force being to accelerate or retard the motion of a body, or to alter its direction, if the tendency could properly be termed a force and the centripetal force which counteracts it were removed, it would accelerate or retard the motion of the body, or alter its direction, which it does not.

The only sense in which the term centrifugal force can be used with propriety as a force may be obtained by the consideration of relative equilibrium, in which case, if the same centripetal force acted on the body, the centrifugal force would keep it in equilibrium, supposing the body were at rest as it would appear to be to an observer moving with it.

Thus, if a body be supported on the surface of the earth, since the body describes a circle about the axis of the earth with uniform velocity, the pressure of the support and the attraction of the earth must have a resultant, whose direction will pass through the centre of this circle, and whose magnitude will be such as would cause the body to describe it; this resultant and the centrifugal force will be in statical equilibrium.

172. In this case of circular motion the force is exerted not in accelerating or retarding the motion, but in changing its direction.

Thus, referring to the figure of Prop. I., if the direction of the impulse at B bisect the angle ABC, the triangle CBc will be isosceles, and $BC = Bc = AB$; therefore the velocities in BC and AB will be equal, and the effect of the impulse has been to change the direction without altering the velocity of the body.

Hence, the regular polygon inscribed in a circle, centre S, can be described with uniform velocity under the action of impulses tending to the centre; and, by similar triangles SBC, CBc,

$$Cc : BC :: BC : BS.$$

And if V be the uniform velocity in the polygon, T the time in a side BC, $BC = V \cdot T$; therefore $Cc = \dfrac{V^2 T^2}{BS}$.

If now the number of sides be indefinitely increased, Cc will be ultimately twice the space through which the body will be drawn from the tangent by the continuous force, see Art. 146; therefore $\dfrac{Cc}{T^2} = \dfrac{V^2}{BS}$ will be the measure of the accelerating effect of the centripetal force tending to the centre of the circle.

Illustrations of Circular Motion.

(1) *A small body is attached by an inelastic string to a point on a smooth horizontal table, to determine the tension of the string when the body describes a circle.*

If the body be set in motion by a blow perpendicular to the string, the string will remain constantly stretched, and the only force which acts on the body in the horizontal plane being in the direction of the fixed point, the areas described round this point will be proportional to the time, and the body will move in a circle with uniform velocity.

Let v be the velocity of projection, and l the length of the string, then the accelerating effect of the tension of the string is $\dfrac{v^2}{l}$; that is, $\dfrac{v^2}{l}$ is the velocity which would be generated in an

unit of time from rest by the action of this tension continued constant, therefore the tension of the string : the weight of the body $:: \dfrac{v^2}{l} : g$.

Ex. If a velocity of two feet a second be communicated perpendicular to a string whose length is a yard,

$$v^2 : lg :: 4 : 3 \times 32 :: 1 : 24,$$

hence the tension is $\dfrac{1}{24}$ th of the weight, and the time of revolution is evidently $\dfrac{2\pi l}{v}$ seconds $= \dfrac{6\pi''}{2} = 9\cdot 4''$, nearly.

(2) *If a particle be attached by a string of given length to a point in a rough horizontal plane, and a given velocity be communicated to it, perpendicular to the string supposed tight, find the tension of the string at any time, the time in which it will be reduced to rest, and the whole arc described.*

Let V be the velocity of projection, l the length of the string in feet, v the velocity at any time t. Since the particle describes a small arc ultimately with uniform velocity the accelerating effect of the tension at the time t is $\dfrac{v^2}{l}$. Again, if μ be the coefficient of friction, the retarding effect of friction is μg, which is constant, hence the velocity destroyed in the time t since friction is the only force acting in the direction of the tangent is $\mu g t$, and $v = V - \mu g t$.

Therefore the particle comes to rest in $\dfrac{V}{\mu g}$ seconds after describing the arc $\dfrac{V^2}{2\mu g}$ feet.

The tension of the string at the time t : the weight of the particle $:: \dfrac{v^2}{l} : g :: \dfrac{(V - \mu g t)^2}{l} : g$; therefore the tension $\propto \left(\dfrac{V}{\mu g} - t \right)^2$ $\propto$ the square of the time which will elapse before the particle comes to rest.

(3) *Supposing that the Moon describes a circle with uniform velocity about the centre of the Earth as its centre, to find the ratio of the centripetal acceleration of the Moon's motion to gravity at the Earth's surface.*

z

Let $n =$ number of seconds in the Moon's periodic time, $R =$ the radius of the Moon's orbit in feet; therefore the velocity of the Moon is $\dfrac{2\pi R}{n}$ and $\dfrac{1}{R} \cdot \left(\dfrac{2\pi R}{n}\right)^2$ is the measure of the accelerating effect of the force exerted on the Moon, and the measure of the same for gravity at the Earth's surface $= 32.2$; hence, the ratio required is $4\pi^2 R : 32.2 n^2$.

(4) *A body is suspended by a string from a fixed point, and being drawn out of the vertical is projected horizontally so as to describe a horizontal circle with uniform velocity. Find the velocity and the tension of the string.*

Let A be the point of suspension, BC the radius of the circle described; therefore, the circle being described uniformly, the resultant force on the body tends to the centre B, and the measure of the accelerating effect of this resultant force is $\dfrac{V^2}{BC}$, in the direction CB. Let T, W be the tension of the string and the weight of the body, acting in CA and parallel to AB respectively, therefore $T : W :: CA : AB$;
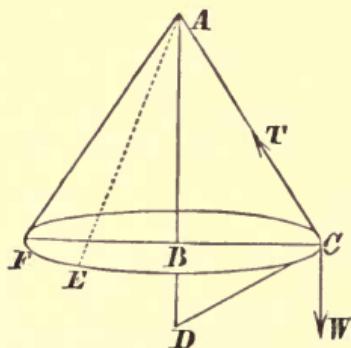
also, $\dfrac{V^2}{BC} : g :: CB : AB$, Art. 171, $\therefore V^2 = \dfrac{g \cdot BC^2}{AB}$,

and, if CD be perpendicular to AC, $BC^2 = AB \cdot BD$; and the velocity will be that due to falling through the space $\frac{1}{2}BD$.

XXI.

1. If the cube of the velocity, in circles uniformly described, be inversely proportional to the periodic time, shew that the law of force will vary inversely as the square of the radii.

2. Compare the areas described in the same time by the planets, supposed to move in circular orbits about the Sun in the centre exerting a force which varies inversely as the square of the distance.

3. If the forces by which particles describe circles with uniform velocity vary as the distance, shew that the times of revolution will be the same for all.

4. If the velocity of the Earth's motion were so altered that bodies would have no weight at the equator, find approximately the alteration in the length of a day, assuming that, before the alteration, the centrifugal force on a body at the equator was to its weight :: 1 : 288.

5. A particle moves uniformly on a smooth horizontal table, being attached to a fixed point by a string, one yard long, and it makes three revolutions in a second. Compare the tension of the string with the weight of the particle.

6. A body moves in a circular groove under the action of a force to the centre, and the pressure on the groove is double the given force on the body to the centre, find the velocity of the body.

7. If a locomotive be passing a curve at the rate of twenty-four miles an hour, and the radius of the curve be $\frac{11}{8}$ of a mile, prove that the resultant of the forces which retain it on the line, viz. of the action of the rails on the flanges of the wheels, and the horizontal part of the forces which act perpendicular to the inclined road-way, will be $\frac{1}{100}$ of the weight of the locomotive, nearly.

8. If a body be attached by an extensible string to a fixed point in a smooth horizontal table, find the velocity with which the body must move in order to keep the string constantly stretched to double its length.

If W be the weight of the body, and nW be the weight which if suspended at the extremity of the string would just double its length, l the length of the string, shew that the square of the required velocity $= 2nlg$.

9. A man stands at the North Pole and whirls 24 lbs. troy weight on a smooth horizontal plane by a string a yard long at the rate of 100 turns a minute; he finds that the difference of the forces which he has to exert according as he whirls it one way or the opposite is roughly 39 grains; find the period of the rotation of the earth.

10. Two equal bodies lie on a rough horizontal table, and are connected by a string which passes through a small ring on the table; if the string be stretched, find the greatest velocity with which one of the bodies can be projected in a direction perpendicular to its portion of the string without moving the other body.

PROP. V.   PROBLEM I.

Having given the velocity with which a body is moving at any three points of a given orbit, described by it under the action of forces tending to a common centre, to find that centre.

Let the three straight lines PT, TQV, VR, touch the given orbit in the points P, Q, R respectively, and let them meet in T and V.
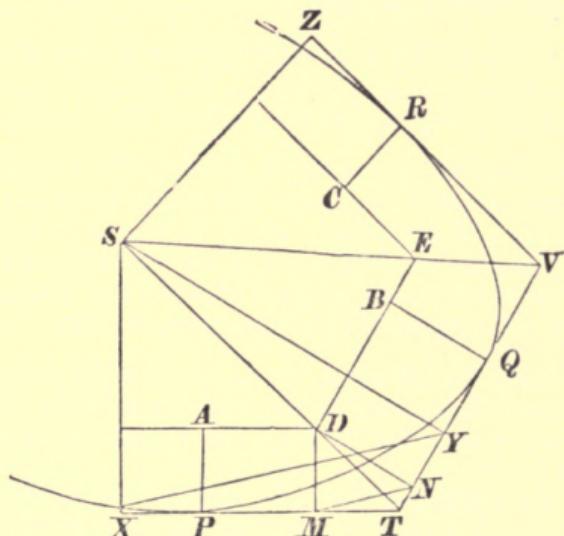
Draw PA, QB, RC perpendicular to the tangents, and inversely proportional to the velocities of the body at the points P, Q, R.   Through A, B, C draw AD, DBE, CE at right angles to PA, QB, RC meeting in D and E.   Join TD, VE; TD and VE produced, if necessary, shall meet in S the required centre of force.

For, the perpendiculars SX, SY, let fall from S on the tangents PT, TQV, are inversely proportional to the velocities at P, Q (Prop. I. Cor. 1), and are therefore directly as the perpendiculars AP, BQ, or as the

perpendiculars DM, DN on the tangents.   Join XY, MN, then, since $SX : SY :: DM : DN$ and the angles XSY, MDN are equal, therefore the triangles SXY, MDN are similar; therefore $SX : DM :: XY : MN :: XT : MT$, and the angles SXT, DMT are right angles; therefore, S, D, T are in the same straight line. Similarly S, E, V are in the same straight line, and therefore the centre S is the point of intersection of TD, VE.   Q. E. D.

XXII.

1. If AB, BC, CD, the three sides of a rectangle, be the directions of the motion of a body at three points of a central orbit, and the velocities be proportional to these sides respectively, prove that the centre of force will be in the intersection of the diagonals of the rectangle.

2. If the velocities at three points of a central orbit be respectively proportional to the opposite sides of the triangle formed by joining the points, and have their directions parallel to the same sides, prove that the centre of force will be the centre of gravity of the triangle.

3. Three tangents are drawn to a given orbit, described by a particle under the action of a central force, one of them being parallel to the external bisector of the angle between the other two.   If the velocity at the point of contact of this tangent be a mean proportional between those at the points of contact of the other two, prove that the centre of the force will lie on the circumference of a certain circle.

4. If the velocities be inversely proportional to the sides of the triangle formed by the tangents at the three points, the centre of force will be the point of concourse of the straight lines joining each an angular point of this triangle to the intersection of the tangents to its circumscribing circle at the ends of the opposite side.

5. If the velocity of a particle describing an ellipse under the action of a centre of force vary as the diameter parallel to the direction of its motion directly, and as its distance from one of the axes inversely, prove that the centre of force will be at an infinite distance.

PROP. VI. THEOREM V.

*If a body revolve about a fixed centre of force, in any orbit
whatever, in a non-resisting medium, and if, at the ex-
tremity of a very small arc, commencing from any point
in the orbit, a subtense of the angle of contact at that point
be drawn parallel to the radius from that point to the
centre of force, then the force at that point tending to the
centre will be ultimately as the subtense directly and the
square of the time of describing the arc inversely.*

Let PQ be the small arc, PS the radius drawn from P
to S, the centre of force. RQ the subtense of the
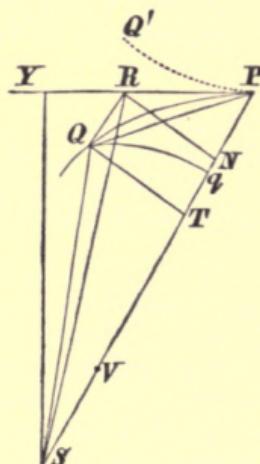
angle of contact at P, parallel to PS. T the time of
describing PQ. F the accelerating effect of the
force at P.

Then, when the body leaves P, it would, if not acted on
by the central force, move in the direction PR, and if
the force F continued constant in magnitude and
direction throughout the time T, QR would be the
space through which it would have been drawn by F
in that time; therefore ultimately, $F = \dfrac{2QR}{T^2} \propto \dfrac{QR.}{T^2}$

COR. 1. Draw QT perpendicular to SP, and let $h =$ twice

the area described in an unit of time. Then area $PSQ = \frac{1}{2}ht$, Prop. I., also, since triangle $PSQ = \frac{1}{2}SP.QT$, and area $PSQ =$ triangle PSQ, ultimately, Lemma VIII., therefore $hT = SP.QT$, ultimately ;

hence, ultimately, $F = 2\dfrac{QR}{T^2} = \dfrac{2h^2}{SP^2}\cdot\dfrac{QR}{QT^2}$.

Cor. 2. Draw SY perpendicular on PR. Then, $\triangle PSQ = \triangle PSR = \frac{1}{2}SY.PR$;

$\therefore hT = SY.PR = SY.PQ$, ultimately ;

hence, ultimately, $F = 2\dfrac{QR}{T^2} = \dfrac{2h^2}{SY^2}\cdot\dfrac{QR}{PQ^2}$.

Cor. 3. If the orbit have finite curvature at P, and PV be the chord of the circle of curvature whose direction passes through S, $PV.QR = PQ^2$, ultimately ;

$$\therefore F = \frac{2h^2}{SY^2.PV}.$$

Cor. 4. If V be the velocity at P, then $V = \dfrac{PQ}{T}$, and

$F = \dfrac{2QR}{T^2} = \dfrac{2QR}{PQ^2}\cdot\left(\dfrac{PQ}{T}\right)^2$, ultimately ;

$$\therefore F = \frac{2V^2}{PV}, \text{ or } V^2 = 2F.\frac{PV}{4} ;$$

that is, the velocity at any point of a central orbit at which the curvature is finite is that which would be acquired by a body moving from rest under the action of the central force at that point continued constant, after passing through a space equal to a quarter of the chord of curvature at that point drawn in the direction of the centre of force.

Cor. 5. Hence, if the form of any curve be given, and the position of any point S, towards which a centripetal force is continually directed, the law of the centripetal force can be found, by which a body will be deflected from its direction of motion, so as to remain in the curve. Examples of this investigation will be given in the following problems.

Observations on the Proposition.

173. In Newton's enunciation of the proposition, the sagitta of the arc, which bisects the chord and is drawn in the direction of the centre of force, is employed instead of the subtense used in the text, but these are ultimately proportional by Art. 90.

The variations by which Newton expresses the results of the first three corollaries are replaced by equations, in order to facilitate the comparison of the motion of bodies in different orbits and the forces acting upon them.

174. The figure employed in proof of the proposition is drawn upon supposition that the force is attractive, the orbit being concave to the centre of force; the same proof will apply also to the case of a repulsive force, if the curve be drawn in the direction of the dotted line PQ' and the same construction be made.

The exception, however, should be made, that the method fails in the particular positions in which the body is at the points of contact of tangents drawn from the centre of force to the curve; in such cases QR does not ultimately meet the tangent at a finite angle or is not a subtense; the result of the proposition is therefore not demonstrated for these particular positions. A further discussion of the case is given on the next proposition.

175. In the proof it is assumed that the body moves ultimately in the same manner as if the force P remained constant in magnitude and direction, in which case the body would describe a parabola, whose axis is parallel to PS, and which is evidently the parabola which has at P the same curvature as the curve. By this consideration the proposition contained in Cor. 4 can be readily proved. For, since the body moves in a parabola under the action of a constant force in parallel lines, the velocity at P is that acquired by falling from the directrix under the action of the force at P, continued constant, *i.e.* through a space equal to the distance of the focus of the parabola, which is equal to a quarter of the chord of curvature at P, drawn through S.

176. The supposition that the force at P continued constant in magnitude and direction, causes the body to move in a curve which is ultimately coincident with the path of the body, may be justified by considering that if PQ' be the arc of the parabola described on this supposition in the same time as the arc PQ actually described, the error $Q'Q$ is due to the change in the magnitude of the forces and the direction of their action in the two cases; now, the greatest difference of magnitude varies as the difference of SP and SQ ultimately, and the ratio of the error from this cause to $Q'R$ vanishes ultimately; also, since $\angle PSQ$ vanishes ultimately, the ratio of the error, arising from the change of direction, to $Q'R$ vanishes; therefore, $Q'Q : Q'R$ vanishes, and the curves may be considered ultimately coincident.

177. It is evident that the results of the Proposition and of the fourth corollary are true of the resultant of any forces, under the action of which any plane orbit is described, for this resultant may be supposed ultimately constant in direction and magnitude, in which case the curve described is a parabola. Hence, as in Art. 175, if F be the accelerating effect of the resultant of the forces, QR the subtense parallel to the direction of the resultant,

$$V^2 = 2F.\frac{PV}{4}, \text{ and } F = 2 \text{ limit } \frac{QR}{T^2}.$$

Homogeneity.

178. Cor. 1, 2. In the expressions for F obtained in these corollaries, it is of great importance to observe the dimensions of the symbols. Thus hT represents an area and h is of two dimensions in linear space and of -1 in time; therefore $h^2.QR$ is of five in space, and of -2 in time, and $SP^2.QT^2$ of four dimensions in space; hence, $\dfrac{2h^2.QR}{SP^2.QT^2}$ is of one dimension in space and of -2 in time, and represents either twice the space through which a force would draw a body in an unit of time, or the velocity generated by the force in an unit of time, either of which may be taken as the measure of the accelerating effect of the force; moreover, this unit is the same by which the magnitude of h is determined.

A A

Hence, if the actual areas, lines, &c., be represented by the symbols, and not the *number* of units, as mentioned in Art. 168, every term of an equation or of a sum or difference must be homogeneous, or of the same number of dimensions, both in space and time; for example, $PQ + V.T$ representing a line, V must be of -1 dimensions in time.

Tangential and Normal Forces.

179. *To find the accelerating effect of the components of the forces, under the action of which a body describes any plane curve, taken in the directions of the normal and tangent at any point.*

Let PQ be a small arc of the curve described under the action of any forces, T, N the measures of the accelerating effect of these forces, in the direction of the tangent and perpendicular to it. Then, if V be the velocity at P, t the time of describing PQ, the forces may be supposed ultimately to remain constant; therefore, if QR be perpendicular to PR, we shall have ultimately $QR = \frac{1}{2}N.t^2$, and $PR = V.t + \frac{1}{2}T.t^2 = V.t$ since the ratio of $T.t^2 : Vt$ vanishes ultimately; hence, if ρ be the radius of curvature at P, $2\rho = \dfrac{PR^2}{QR} = \dfrac{2V^2}{N}$ ultimately; therefore $\dfrac{V^2}{\rho}$ will be the measure of the normal acceleration estimated towards the centre of curvature.

Again, if V' be the velocity at Q, V' will be ultimately the component of the velocity in the direction PR; therefore, by Art. 53, we obtain two measures of the tangential acceleration, the limits of $\dfrac{V'^2 - V^2}{2PQ}$ and $\dfrac{V' - V}{t}$.

180. *To find the velocity at any point of an orbit described under the action of any forces in one plane.*

Let AB be any arc of an orbit, V, v the velocities at A and B, and suppose the arc AB divided into a large number of small portions, of which PQ is one, v_r, v_{r+1} velocities at P and Q, T the accelerating effect of the tangential component of the forces at P,

$$v_{r+1}^2 - v_r^2 = 2T.PQ \text{ ultimately,}$$

and $v^2 - V^2$ is obtained by taking the limit of the sum of the

magnitudes $2T.PQ$ corresponding to the different arcs when their number is indefinitely increased.

That this is rigidly correct may be shewn by considering that $v_{r+1}^2 - v_r^2 : 2T.PQ$ is ultimately a ratio of equality; therefore, by Cor., Lemma IV., or Art. 22, the limiting ratio of the sums is also a ratio of equality.

In the case of a central force, whose accelerating effect is F, $T = F \cos RPS$;

$$\therefore\ v_{r+1}^2 - v_r^2 = 2F.PQ \cos RPS = 2F(SP - SQ)\ \text{ultimately,}$$

whence $v^2 - V^2$, if F depend only on the distance.

Radial and Transversal Forces.

181. *To find the accelerating effect of the components of force, under the action of which a body describes any plane curve, taken in the direction of a radius vector drawn from a fixed point, and perpendicular to it.*

Let PQ be a small arc described in the time T; QRU, PU parallel and perpendicular to SP; P, Q the measures of the accelerating effects of the components in PS and PU; PR a
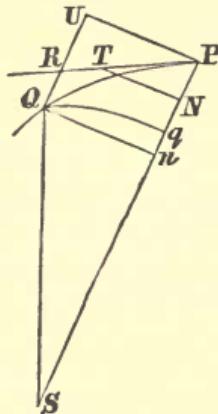
tangent at P. If V be the velocity at P, make $PT = V.T$, draw TN perpendicular to SP, and let Qq be the arc of a circle, centre S.

Since the forces may be considered ultimately constant in magnitude and direction, $\frac{1}{2}P.T^2 = Nn = Nq + \dfrac{Qn^2}{2Sq}$ ultimately.

Let h be twice the area which would be described in an unit of time by radii from S, if the transverse force Q ceased to act, then $Qn.SP = TN.SP = h.T$; therefore $\dfrac{Qn^{x}}{2\,Sq} = \dfrac{h^{x}T^{x}}{2\,SP^{3}}$ ultimately; and if P' be the measure of the accelerating effect of a force, under the action of which the body would move in PS, so that its distance from S would be always equal to that of the body in PQ at the same time, $\frac{1}{2}P'.T^{2} = Nq$ ultimately; therefore $P = P' + \dfrac{h^{2}}{SP^{3}}$.

Again, if at Q h' correspond to h, $h' - h$, the increase of h, will be due to the increase of velocity in direction PU, which is equal to $Q.T$ ultimately; therefore $(h' - h)\,T = Q.T^{x}.SP$ ultimately; hence $Q = \dfrac{h' - h}{SP.T}$ ultimately.

Angular Velocity.

182. DEF. *Angular velocity* of a point moving about a fixed point is the rate at which angles are described by radii drawn to the fixed point.

Uniform angular velocity is measured by the angle described in an unit of time.

Variable angular velocity is measured by the angle which would be described by a radius in an unit of time, if moving with uniform angular velocity equal to the angular velocity at the time under consideration; this is the limit of the angle, described in a time T, divided by T, when T is indefinitely diminished.

183. *To find the angular velocity in a central orbit.*

Let PQ be a small arc described in the time T, draw QN perpendicular to SP, then $h.T =$ twice the area $PSQ = QN.SP$ ultimately; and, if the angles be supposed estimated in circular measure, $\angle PSQ = \dfrac{QN}{SQ} = \dfrac{QN}{SP}$ ultimately; therefore the angular velocity, which is $\angle \dfrac{PSQ}{T}$ ultimately, $= \dfrac{h}{SP^{2}}$.
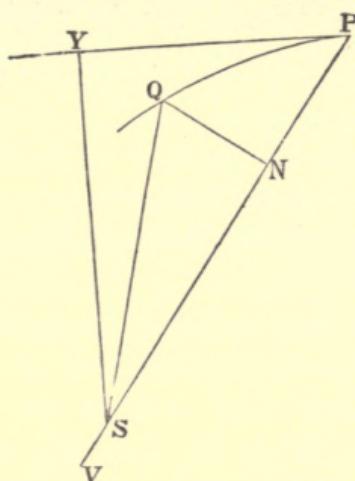
184. *To find the angular velocity of the perpendicular on the tangent from the centre of force.*

Draw SY perpendicular on the tangent PY, and let PV be the chord of curvature through S.

The angle described by SY in the time T is equal to the angle between the tangents at P and Q, or to twice the angle PVQ; therefore angular velocity of SY : angular velocity of $SP :: 2 \angle PVQ : \angle PSQ :: 2SQ : QV$ ultimately; hence the angular velocity of $SY = \dfrac{2h}{PV.SP}$.

Illustrations.

(1) *To find the tension of a string by which a body is attached to the centre of a vertical circle in which it revolves.*

Let P be the position of the body at any time, CP, CA radii drawn to P and the lowest point, and let v, u be the velocities at P and A. Draw PM perpendicular to CA. Then $u^2 - v^2 = 2g.AM$ and $\dfrac{v^2}{CA}$ is the accelerating effect of the forces in the direction PC, viz. the tension of the string and the component of the weight of the body. Let T be the tension of the string and m the mass of the body;

$$\therefore \frac{v^2}{CA} = \frac{T}{m} - g.\frac{CM}{CP} \text{ and } T : mg :: v^2 + g.CM : g.CA;$$

therefore the tension of the string : the weight of the body

$$:: u^x - 2g.CA + 3g.CM : g.CA.$$

NOTE 1. In order that the complete circle may be described, since the string must be stretched at the highest point where $-CA$ must be written for CM, $u^x =$ or $> 5g.CA$, and if the circle be just described, the tension at the lowest point will be six times the weight.

NOTE 2. If the body oscillate, the extent of the oscillation will be given by the consideration that at the extremity P' of the arc of oscillation there will be no velocity, therefore $u^x = 2g.AM'$, and AM' is less than AC, otherwise the string would not be stretched, so that the tension at A : the weight $:: 2AM' + AC : AC.$

(2) *Find the force under the action of which a body may describe the equiangular spiral uniformly.*

The velocity being constant, there is only a normal force measured by (vel.)$^2 \div$ radius of curvature $= \dfrac{V^2 \sin \alpha}{SP}$, Art. 128.

(3) *Find the force tending to the pole of the cardioid, under the action of which the curve is described.*

Since $PV = \frac{4}{3}SP$, and (vel.)$^2 = \dfrac{h^2}{SY^2} = \dfrac{h^2.BC}{SP^3}$, see page 105,

therefore the accelerating effect of the force is $\dfrac{3h^2.BC}{2SP^4} \propto \dfrac{1}{SP^4}$.

(4) *Two equal rings P, Q slide on a string which passes round two fixed pegs A, B in a smooth horizontal plane ; the rings are brought together, and then projected with equal velocities, so as to keep the string stretched symmetrically. Shew that the tension of* he *string varies inversely as the distance AP.*
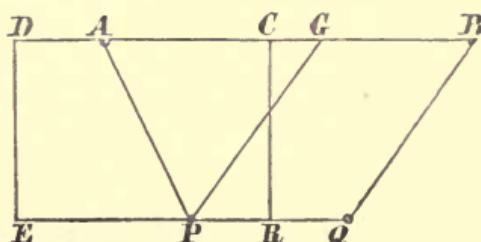
The figure represents the position of the system at any time. Let CR bisect AB and PQ, and let DE be drawn parallel to CR, so that $EP = PA$, then $EPR = AP + PR$ is constant; therefore DE is fixed, and P moves in a parabola whose focus is A and directrix DE.

Also, the tensions of the string in PA, PQ being equal, and equally inclined to the tangent to P's path, the resultant of these tensions, which are the only forces acting in the plane of the curve, acts in the normal, hence the rings move with uniform velocity equal to the velocity of projection V, and if T be the measure of the accelerating effect of the tension, PG the normal, ρ the radius of curvature, $2T \cos APG = \dfrac{V^2}{\rho}$, and $2\rho \cos APG$ = chord of curvature through $A = 4PA$; therefore

$$T = \frac{V^2}{4PA} \propto \frac{1}{PA}.$$

(5) *A body revolves in a smooth circular tube under the action of a force tending to any point in the circumference, and varying as the distance from that point. Find the pressure on the tube, and the point where there is no pressure, the motion commencing from a given point.*

Take A the centre of force, C that of the circle; let B be the point of starting, PQ a small arc, BD, PM, QN ordinates to the
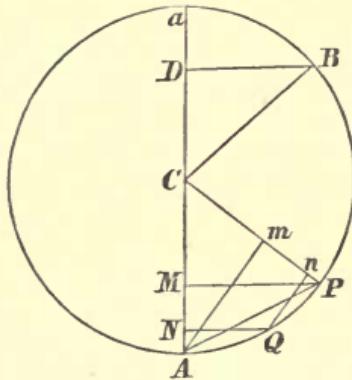
diameter through the centre of force, Am, Qn perpendicular on CP; let $\mu . PA$ be the measure of the accelerating effect of the force at P; therefore $\mu . mA$, $\mu . Pm$ are those of the tangential and normal forces, $= \mu . PM$ and $\mu . AM$ respectively.

(vel.)² at $Q-$(vel.)² at $P=2\mu.PM.PQ=2\mu.CP.MN$ ultimately, see Art. 179, whence, taking the limit of the summation for all the small arcs in BP, (vel.)² at $P=2\mu.CP.DM$.

Also, $\dfrac{(\text{vel.})^2 \text{ at } P}{CP} = \mu.AM \mp$ the accelerating effect of the pressure of the tube, the upper or lower sign being taken according as the pressure is from or towards C; therefore the pressure on the tube has for the measure of its accelerating effect

$$\pm\mu\,(AM-2DM)=\pm\,(3AM-2AD);$$

hence the pressure is outwards from B until $AM=\frac{2}{3}AD$, at which point there is no pressure, and inwards from that point to the corresponding one on the opposite side, having its greatest value at A, and the outward pressure at B is half the inward pressure at A.

(6) *If in a smooth elliptic tube a particle be placed at any point, and be acted on by two forces which tend to the foci and vary inversely as the square of the distances from those points, shew that the pressure at any point will vary as the curvature.*

Let O be the point of starting, PQ a small arc described by the body, QT, QU perpendiculars on SP, HP.

Take $\dfrac{\mu}{SP^2}$, $\dfrac{\mu'}{HP^2}$, R, as the measures of the accelerating effects of the forces, and of the pressure of tube outwards.

Then, employing the usual letters for the lines of the figure, the accelerating effect of the tangential component of force to S is

$$\frac{\mu}{SP^2} \cdot \frac{PT}{PQ} = \frac{\mu\,(SP-SQ)}{SP.SQ.PQ} = \frac{\mu}{PQ.SQ} - \frac{\mu}{PQ.SP}\ \text{ultimately};$$
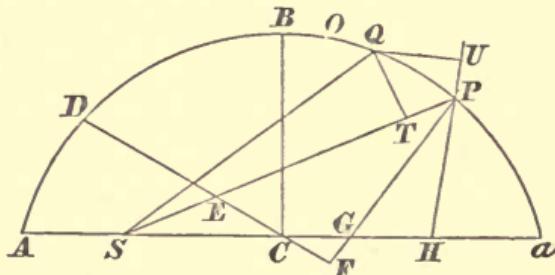
and similarly for the force tending to H;

$\therefore$ (vel.)^2 at $P -$ (vel.)^2 at $Q = \left(\dfrac{2\mu'}{HP} - \dfrac{2\mu'}{HQ}\right) - \left(\dfrac{2\mu}{SQ} - \dfrac{2\mu}{SP}\right)$;

$\therefore$ (vel.)^2 at $P = \dfrac{2\mu'}{HP} - \dfrac{2\mu'}{HO} - \dfrac{2\mu}{SO} + \dfrac{2\mu}{SP}$.

Also, $\dfrac{\text{(vel.)}^2 \text{ at } P}{\rho} = \left(\dfrac{\mu'}{HP^2} + \dfrac{\mu}{SP^2}\right) \dfrac{PF}{PE} - R$, if ρ be the radius

of curvature at P, and $2\rho \cdot \dfrac{PF}{PE} = PV = \dfrac{2CD^2}{AC} = \dfrac{2SP.HP}{AC}$;

$\therefore R.\rho = \dfrac{\mu'.SP}{AC.HP} + \dfrac{\mu.HP}{AC.SP} - \dfrac{2\mu'}{HP} - \dfrac{2\mu}{SP} + \dfrac{2\mu'}{HO} + \dfrac{2\mu}{SO}$

$= \dfrac{2\mu'}{HO} + \dfrac{2\mu}{SO} - \dfrac{\mu'}{AC} - \dfrac{\mu}{AC} = \left(\dfrac{\mu'SO}{HO} + \dfrac{\mu HO}{SO}\right)\dfrac{1}{AC}$,

which is constant; therefore R varies as the curvature.

XXIII.

1. A body is attached to a point by a thread, and is projected so as to describe a vertical circle, prove that, if T_1, T_2 be the tensions of the string at the extremities of any diameter, the arithmetic mean between T_1, T_2 is independent of the position of the diameter, and that $T_2 \sim T_1$ is six times the component of the weight in the direction of the diameter.

2. A string of given length l is capable of sustaining a weight W. One end is fixed, and a given weight P less than W, attached to the other end, oscillates in a vertical plane, find the greatest arc through which the weight can oscillate without breaking the string.

3. A ring slides on a string hanging over two pegs in the same horizontal line, find the tension of the string at the lowest point, if the ring begin to fall from the point in the horizontal line through the pegs, the string being stretched.

4. AB is the vertical axis of a cycloid, A the highest point, AM, AN are the abscissæ of points at which a body begins to slide down the arc of the cycloid, and at which it leaves the curve; prove that N is the middle point of MB.

5. If in a central orbit the direction of motion change uniformly, prove that the normal force will vary as the radius of curvature.

6. Given the Sun's motion in longitude at apogee and perigee to be 57' 10" and 61' 10"; find the eccentricity of the Earth's orbit, supposed to be an ellipse about the Sun in one of the foci.

7. Prove that the angular velocity of a projectile about the focus of its path varies inversely as its distance from the focus.

8. A particle, constrained to move on an equiangular spiral, is attracted to the pole by a force proportional to the distance, prove that, at whatever point the particle be placed at rest, the times of describing a given angle about the centre of force will be the same.

9. A body slides down a smooth cycloidal arc, whose axis is vertical and vertex downwards, find the pressure at any point of the cycloid, and shew that, if it fall from the highest point, the pressure at the lowest point will be twice the weight of the body.

10. Find the law of force, tending to the centre, under the action of which a lemniscate can be described.

XXIV.

1. Two straight lines AB and BC are united at B; AB revolves about A, and BC about B with the same uniform angular velocity; shew that the acceleration on C tends to A and varies as CA.

2. A particle describes an ellipse, the centre of force being situated at any point within the figure. Shew that at the point where the *true* angular velocity is equal to the *mean* angular velocity, the radius vector is a mean proportional between the semiaxes.

3. A particle begins to move from any point of a smooth parabolic tube, being attracted to the focus by a force which varies inversely as the square of the distance; find the greatest pressure.

4. If SY be the perpendicular on the tangent at a point P of an orbit, described about a centre of force S, prove that the acceleration at P will be equal to the product of the velocities of P and Y divided by SY.

5. A smooth cone is placed with its axis vertical and vertex upwards, shew that there is a certain portion of the surface upon which a particle can describe a circle, if properly projected and acted on by gravity and by a force tending to the vertex and varying as the distance.

6. Shew that the force required for the description of an ellipse about the vertex A varies as $\dfrac{AP}{AN^3}$, where PN is the perpendicular on the axis.

7. If a particle describe an ellipse under the action of a force tending to any fixed point O, the force will vary as $\dfrac{DD'^3}{OP^2 . PP'^3}$, where P is the position of the particle, PP' the chord through O, and DD' the diameter parallel to this chord.

8. Shew that in the elliptic orbit described under the action of a force tending to a focus, the angular velocity round the other focus varies inversely as the square of the diameter parallel to the direction of motion.

9. A particle moves in a circular tube, under the action of a force which tends to a point in the tube, and whose accelerating effect varies as the distance, shew that, if the particle begin to move from a point at a distance from the centre of force equal to the radius, there will be no pressure on the tube at an angular distance from the centre of force equal to $\cos^{-1}\frac{1}{3}$.

10. A particle moves in a smooth elliptic groove, under the action of two forces tending to the foci and varying inversely as the squares of the distances, the forces being equal at equal distances. Prove that, if the velocity at the extremity of the axis major be to that at the extremity of the axis minor as AC to BC, then the velocity at any point will vary inversely as the normal; find the pressure on the tube.

11. Determine the relation between μ and λ and the velocity of projection, in order that an ellipse may be described under the action of forces $\dfrac{\mu}{SP^2}$, $\dfrac{\mu}{HP^2}$ to the foci and $\lambda . CP$ to the centre, acting simultaneously.

12. A particle is attached to a point C by a string, and is attracted by a force which tends to a point S, and varies inversely as the square of the distance from S. Find the least velocity with which the particle can be projected from a point in CS, or CS produced, so as to describe a complete circle. If CS be less than the length of the string, prove that the tension will be a maximum at a point D, where SD is perpendicular to CS, and that if CS be half the length of the string, the two minimum and the maximum tensions will be as 0, 4 and $3\sqrt{3}$.

PROP. VII.   PROBLEM II.

A body moves in the circumference of a circle, to find the law of the centripetal force, tending to any given point in the plane of the circle.

Let APV be the circumference of the circle, S the given point to which the centripetal force tends, PV the
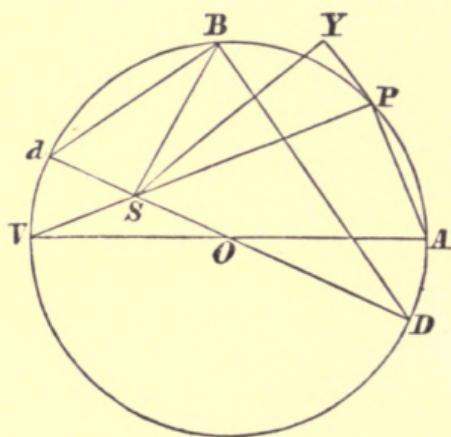
chord of the circle drawn through S from P, the position of the body at any time, and VOA the diameter through V.   Join PA, and draw SY perpendicular to PY, the tangent to the curve at P.

By Prop. VI. Cor 3, if F be the measure of the accelerating effect of the centripetal force, $F = \dfrac{2h^2}{SY^2.PV}$,

and, since the angles SPY, VAP are equal, and also the right angles PYS, APV, the triangles SPY, VAP are similar, and $SY : SP :: PV : VA$;

$$\therefore F = \frac{2h^2.VA^2}{SP^2.PV^3};$$

therefore, since h and VA are given, F varies inversely as $SP^2.PV^3$.

COR. 1. Hence, if the given point S to which the centripetal force tends, be situated on the circumference of the circle, V will coincide with S, and F vary inversely as SP^5.

COR. 2. The force, under the action of which a body P revolves in a circle $APTV$, is to the force, under the action of which the same body P can revolve in the same circle in the same periodic time about any other centre of force R, as $RP^2.SP$ to SG^3, SG being a straight line drawn from the first centre S, parallel to the distance RP of the body from the second centre of force R, to meet PG, a tangent to the circle.

For, by the construction of this proposition, since the periodic times are the same, the areas described in
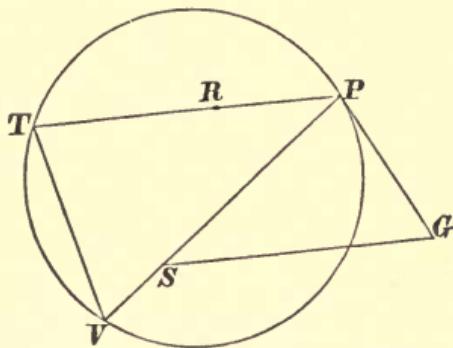
a given time are the same; therefore, h is the same for both centres, hence, if PRT be the chord through R, the force to S : the force to $R :: \dfrac{RP^2.PT^3}{SP^2.PV^3}$; but, by similar triangles TPV, GSP, $PT : PV :: SP : SG$; therefore force to S : force to R $:: RP^2.SP^3 : SP^2.SG^3 :: RP^2 SP : SG^3$.

COR. 3. The force, under the action of which a body P revolves in any orbit about a centre of force S, is to the force, under the action of which the same body P can revolve in the same orbit in the same periodic time about any other centre of force R, as

$RP^2.SP$ to SG^3, SG being the straight line drawn from
the first centre of force S, parallel to RP the distance
of P from the second centre of force R, to meet PG
the tangent to the orbit.

For, in each case, the body may be supposed for a
short time to be moving in the circle of curvature,
and the forces are the same as those which would
retain the body in the circular orbit; therefore,
since the areas described in a given time are equal,
the ratio of the forces is $RP^2.SP : SG^3$.

Observations on the Proposition.

185. In the figure employed in the proposition, the force is
supposed to be attractive, but the investigation of the law
of force applies also to the case in which the centre of force
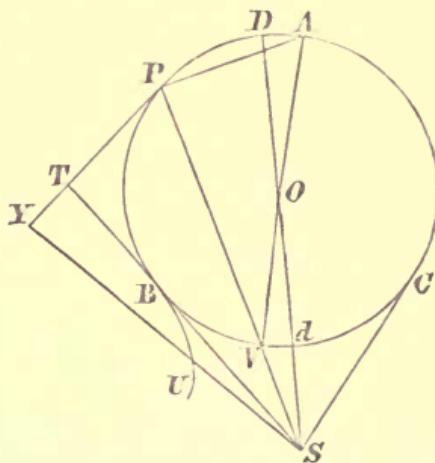
S is exterior to the circle, in which case the force is repulsive
through the arc BC, which is convex to the centre of force, and
contained between the tangents drawn from S to the circle.

It is important, however, to observe that this problem is to
find what would be the law of force tending to S, under the
action of which a body would be moving, supposing that it
could move in the circle, or any portion of the circle, under the
action of such a force, but it does not assert the possibility of
such a motion, which is considered in Art. 165.

In fact, the *complete* description of a circle ABC, under the sole action of a central force tending to an external point S, is impossible, because, as the body approaches the point B, the component of the velocity perpendicular to SB remains finite however near the body approaches B, and since there is no force to generate a velocity in the opposite direction, the body must proceed to describe an arc BU on the opposite side. SB would be a tangent to both curves, because the velocity in direction BS becomes larger than any finite quantity, as the body approaches B, and therefore the angle between BS and the direction of motion is indefinitely small at B.

That a finite velocity in the direction perpendicular to SB could remain up to B, may be shewn by producing SB to T in the tangent PY at P; then the component of the velocity at P perpendicular to SB is $\dfrac{h}{SY} \cdot \dfrac{SY}{ST} = \dfrac{h}{ST} = \dfrac{h}{SB}$, when the body arrives at a point very near to B.

186. The force at a point indefinitely near to B cannot be properly determined by the method of Prop. VI., because the lines parallel to the direction of the force from which the mea-

sures of the force are obtained are not subtenses, or sagittæ, since they are in this case not inclined at a finite angle to the tangent.
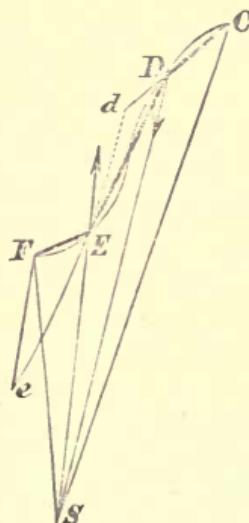
But it can be seen in another manner from the polygon of Prop. I, that the force is infinitely great, when the distance from B becomes infinitely small.

Thus, if $CDEF$ be a portion of the polygon whose limit touches the radius from S between D and E, the angle between DE and DS or ES may be made as small as we please compared with the angle between CD and DE, hence the velocity generated by the impulse in the directions DS and SE will become infinitely great compared with the velocities in CD and EF. In the figure, the impulses at D and E, whose directions are denoted by the arrows, have corresponding to them in the limit the forces on opposite sides of the tangent, which are attractive and repulsive respectively.

187. COR. 1. For the reasons given above, a limitation should be made, viz., when P is at a finite distance from S. In this case $PV = SP$ and $F = \dfrac{8h^2R^2}{SP^5}$, R being the radius of the circle.

We may also observe here that the *possibility* of a description of a circle is not asserted, but only the law of force required *in case of* description of any portion of the circle. The complete description of the single circle is, in fact, impossible, for, under the action of the force obtained, the body would pass to the other side of the tangent on arriving at S, then proceed to describe another equal circle, and, on arriving again at S, return into the original circle.

188. COR. 2. The orbit being the same, and also the periodic times about S and R being equal, the value of h, in the two cases, is the same; also, the force tending to S for the orbit being of the same magnitude at P as that under the action of which the circle of curvature would be described, and SY, PV being the same in the orbit and the circle, h is also the same, Prop. VI. Cor. 3; and, similarly, h is the same in the circle and orbit described about R; therefore it is the same in the circle described about S and R as centres of force, and hence Cor. 2 applies.

Absolute Force.

189. If the force upon a body placed at any distance from the point S vary inversely as the nth power of that distance, the magnitude of the force, or its ratio to any given force, as that of gravity, will be determined when the distance SP is given. The measure of the accelerating effect of the force is written $\frac{\mu}{SP^n}$, where μ the constant part of this measure is an algebraical symbol of $n+1$ dimensions in linear space. If the unit of space $= a$, $\frac{\mu}{a^n}$ is the measure of the accelerating effect of the force on a body at an unit of distance, and μ is called the *Absolute Force*, being the measure of the accelerating effect of the force at an unit of distance $\times$ the nth power of that unit. The absolute force is not the measure of the accelerating effect of any force, unless the symbols be treated numerically, in which case μ is twice the number of units of space through which a constant force, equal to the force at an unit of distance, would draw a body from rest in an unit of time.

Law of Force in a Circular Orbit.

190. The law of force may be expressed in terms of the distance SP, for SD, Sd being the greatest and least distances of the body from S, $SD.Sd = SP.SV$; see figure, page 188.

$$\therefore SP.PV = SP^2 \pm SD.Sd,$$

$+$ or $-$ according as S is within or without the circle ;

$$\therefore F = \frac{2h^2.AV^2.SP}{(SP^2 \pm SD.Sd)^3}.$$

If S be on the circumference $Sd = 0$, therefore $F = \frac{2h^2.AS^2}{SP^5}$.

If S be exterior to the circle, $SD.Sd = SB^2$, and the lower sign must be taken; therefore $F = \frac{2h^2AV^2.SP}{(SP^2 - SB^2)^3}$.

Velocity in the Circular Orbit.

191. *To find the velocity in the circular orbit described under the action of a force tending to any point in the plane of the orbit.*

The velocity at $P = \dfrac{h}{SY} = \dfrac{h}{SP} \cdot \dfrac{SP}{SY} = \dfrac{h}{SP} \cdot \dfrac{VA}{PV} \propto \dfrac{1}{SP.PV}$

COR. If S be in the circumference of the circle, and $\dfrac{\mu}{SP}$ be the accelerating effect of the force, $\mu = 2h^2 SA^2$;

hence the velocity at $P = \dfrac{h \cdot VA}{SP^2} = \left(\dfrac{\mu}{2}\right)^{\frac{1}{2}} \cdot \dfrac{1}{SP^2}$.

Or, we may employ the result of Prop. VI., Cor. 4,

$$V^2 = F \cdot \dfrac{PV}{2} = \dfrac{\mu}{SP^5} \cdot \dfrac{SP}{2}; \quad \therefore V = \left(\dfrac{\mu}{2}\right)^{\frac{1}{2}} \cdot \dfrac{1}{SP^2} \propto \dfrac{1}{SP^2}.$$

Periodic Time.

192. *To find the periodic time in a circular orbit described under the action of a force tending to a point in the circumference.*

Let P be the periodic time, R the radius of the circle, and let $\dfrac{\mu}{SP^5}$ be the measure of the accelerating effect of the force at P, then $h.P = $ twice the area of the circle $= 2\pi R^2$,

and $\mu = 2h^2 AS^2 = 8h^2 R^2$; $\therefore P = \dfrac{4\sqrt{2\pi R^3}}{\mu^{\frac{1}{2}}}$.

193. *To compare the periodic times in the same circle when described under the action of a force tending to a point in the circumference, and a force tending to the centre, of the same magnitude as that of the first force at a distance equal to the radius of the circle.*

Let P' be the periodic time, and V the uniform velocity in the circle in the second case, $V^2 = \dfrac{\mu}{R^5} \cdot R$; $\therefore V = \dfrac{\mu^{\frac{1}{2}}}{R^2}$,

and $P' \cdot V = 2\pi R$; $\therefore P = \dfrac{2\pi R^3}{\mu^{\frac{1}{2}}} = \dfrac{P}{2\sqrt{2}}$.

Illustrations.

(1) *When the force in a circular orbit tends to a point within the circle, to find the point at which the true angular velocity is equal to the mean angular velocity.*

The true angular velocity $= \frac{h}{SP^2}$, the mean $= \frac{2\pi}{P} = 2\pi \cdot \frac{h}{2\pi R^2}$; therefore at the required point $SP = R$, or the perpendicular from the required point upon the line joining S to O the centre of the circle bisects OS.

(2) *A body describes a circle under the action of a force tending to a point within it, the measure of whose accelerating effect at the greatest and least distances SD and Sd are the radius and twice the diameter respectively, the unit of time being a second*; *find the number of seconds in passing from D to d.*

$$\text{Since } \frac{8h^2R^2}{SD^2 \cdot Dd^3} = R, \; \frac{8h^2R^2}{Sd^2 \cdot Dd^3} = 4R;$$

$$\therefore SD = 2Sd, \text{ and } 3Sd = Dd = 2R; \; \therefore h = 2R \cdot Sd = \tfrac{4}{3}R^2;$$

and the number of seconds from D to $d = \dfrac{\pi R^2}{h} = \dfrac{3\pi}{4}$.

XXV.

1. If μ be the absolute force in a circular orbit described under the action of a force tending to a point in the circumference, prove that the time in a quadrant commencing from the extremity of the diameter through the centre of force will be $(\pi + 2) R^3 (\tfrac{1}{2}\mu)^{-\frac{1}{2}}$.
   In what unit of time is the result expressed?

2. A point describes a circle, with an acceleration tending to any point within the circle. Prove that, if three points be taken at which its velocities are in harmonical progression, the velocities at the other extremities of the diameters, passing through those points, will also be in harmonical progression.

3. In the case of a centre of force S within a circle, if two points L, M be taken, such that LS, MS make equal angles with the diameter through S, and on the same side of it, then the forces at L and M will be to each other in the inverse ratio of the squares on OL and OM.

4. The sum of the reciprocals of the velocities at the extremities of any diameter is independent of the position of the centre of force, and varies as the periodic time.

5. Prove that, when a circular orbit is described about an internal point, the sum of the square roots of the accelerations at the extremities of any chord passing through that point varies inversely as the square root of the length of the chord.

6. **Prove that, if the law of force tending to** S, a point without a circle, be the law of force under which part of the circle can be described, the body will move near B as if acted on by a force tending to B and varying inversely as the cube of the distance from B.

7. OE is a radius perpendicular to the diameter through S in a circular orbit about a central force tending to a point S within the circle, SB an ordinate, perpendicular to OS, shew that, if the force at B be an arithmetic mean between the forces at the greatest and least distances, $OE^3 = SB . SE^2$.

8. Prove that, if a circle be described about a force tending to a point in the circumference, and PQ be a chord parallel to the diameter through that point, the times of describing equal small arcs near P and Q will differ by a quantity which varies as PQ.

9. When a particle is describing a circle under the action of a central force, shew that at every instant the angular velocities about all points in the circumference are the same.

10. The period in an orbit described under the action of a central force, whose accelerating effect is μr^n is given to be $\lambda a^m \div \mu^{\frac{1}{2}}$, a being a line and λ a number, find n.

11. Apply the proposition contained in Cor. 3, to prove that if in an elliptic orbit described under the action of a force tending to the centre, the force vary as the distance from the centre, then the force tending to the focus will vary inversely as the square of the focal distance.

12. Deduce, by Cor. 3, the law of force, when a parabola is described under the action of a force tending to the focus, from the constant force parallel to the axis, under the action of which the same parabola may be described.

13. Shew, by the method of projections, that the centripetal force at any point P tending to a fixed point O in the axis major of an ellipse under which the ellipse can be described, varies as $\left(\frac{OQ}{PQ}\right)^3 . OP$, POQ being the chord of the ellipse through O.

PROP. VIII.   PROBLEM III.

A body moves in a semicircle PQA under the action of a force tending to a point S so distant that the lines PS, QS drawn from the body to that point may be considered parallel ; to find the law of force.

Let CA be a semidiameter of the semicircle drawn from the centre perpendicular to the direction in which the force acts, cutting PS, QS in M and N, and join CP.
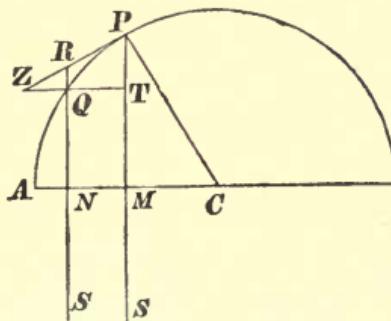
Let PRZ be the tangent at P, ZQT perpendicular to PMS, meeting PRZ in Z, and let SNQ meet PRZ in R.

Then the force at $P = \dfrac{2h^2 QR}{SP^2 . QT^2}$ ultimately, if the arc PQ be indefinitely diminished, and SP may be considered constant; also, by Euclid III. 36,

$$QR.(RN+QN)=RP^2,$$

and, since RQ is parallel to PT, and the triangles PZT, CPM are similar,

$$RP : QT :: ZP : ZT :: CP : PM;$$

$$\therefore \ \frac{QT^2}{QR}=\frac{QT^2}{RP^2}.\frac{RP^2}{QR}=\frac{PM^2}{CP^2}.(RN+QN)$$

$$=\frac{2PM^3}{CP^2} \text{ ultimately};$$

hence force at $P = \dfrac{h^2 CP^2}{SP^2 . PM^3} \propto \dfrac{1}{PM^3}.$

Aliter.

In fig. page 190 draw OE a semidiameter perpendicular to SD, and let the distance SP cut the circle in V, and OE in M, then, by the preceding proposition, $F = \dfrac{8h^2R^2}{SP^2.PV^3}$, and, if S be very distant, the ratio $PM : SM$ or SO will vanish; therefore, $SP = SO$ ultimately, and PV is ultimately perpendicular to OE and equal to $2PM$;

$$\therefore F = \frac{h^2R^2}{SO^2.PM^3} \propto \frac{1}{PM^3}.$$

SCHOLIUM.

A body moves in an ellipse, hyperbola or parabola, under the action of a force tending to a point so situated and so distant that the lines drawn from the body to that point may be considered parallel, and perpendicular to the major axis of the ellipse, the axis of the parabola or the transverse axis of the hyperbola. To shew that the force varies inversely as the cube of the ordinates.

Let AMG be the axis to which the direction of the forces may be considered perpendicular, PM, PG
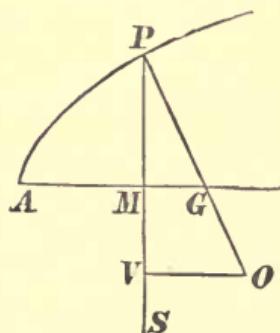
the ordinate and normal, PO the diameter of curvature, and PV the chord of curvature in direction PS.

Then $F = \dfrac{2h^2}{SY^2.PV} = \dfrac{2h^2}{SP^2.PV} \cdot \dfrac{PG^2}{PM^2}$, $\because \dfrac{SP}{SY} = \dfrac{PG}{PM}$,

$\therefore$ $F \propto \dfrac{PG^2}{PM^2.PV} \propto \dfrac{PG^3}{PM^3.PO} \propto \dfrac{1}{PM^3}$;

since $PO \propto PG^3$, Art. 84.

Observations on the Proposition.

194. It has been shewn in Art. 151, that the equable description of areas may, in the case of forces acting in parallel lines, be replaced by the uniformity of the resolved part of the velocity in the direction perpendicular to that of the forces. In the proof given in the text, when S is removed to an infinite distance, h and SP are both infinite magnitudes, but the expression $\dfrac{h}{SP}$ is finite, for area SPQ described in the time T is ultimately equal to area SMN, whose base is equal to uT, u being the component of the velocity perpendicular to the direction of the forces; therefore $hT = uT.SP$ ultimately, and $\dfrac{h^2}{SP^2} = u^2$, hence the acceleration due to the force, when a body describes the semicircle, is $\dfrac{u^2 R^2}{PM^3}$.

195. The accelerating effect of the force, acting in parallel lines, may be obtained directly from the proposition of Art. 151, as follows.

Let u be the constant component of the velocity V, perpendicular to the direction of the force, and let F be the accelerating effect of the force, therefore $F = \dfrac{2V^2}{PV} = \dfrac{V^2}{PM}$;

also $V.u :: CP : PM$; $\therefore$ $F = \dfrac{u^2.CP^x}{PM^3}$.

Extension of Scholium.

196. *When a body describes any curve under the action of a force tending to a point S, so distant that the lines drawn from S*

to the body may be considered parallel; to find the law of force and the velocity at any point.

Let AP be any curve, AMG the line to which the forces are perpendicular, PM, PG the ordinate and normal at the point P, PV the chord of curvature in the direction of the force, PO the diameter of curvature.

Let F be the accelerating effect of the force at P, u the component of the velocity V in the direction AMG;

$$\therefore V : u :: PG : PM,$$

$$\text{also } PV : PO :: PM : PG;$$

$$\therefore F = \frac{2V^2}{PV} = \frac{2u^2 . PG^2}{PM^2 . PO} \cdot \frac{PO}{PV} = \frac{2u^2 . PG^3}{PO . PM^3},$$

$$\text{and the velocity} = u . \frac{PG}{PM}.$$

Illustrations.

(1) *A cycloid is described by a particle, under the action of a force acting in a direction parallel to the axis; find the acceleration and the velocity at any point.*

In the cycloid $PO = 4PG$, and $PM . AB = PG^2$, AB being the length of the axis;

$$\therefore F = \frac{2u^2 . PG^2}{PM^3} \cdot \frac{PG}{PO} = \frac{u^2 . AB}{2PM^2} \propto \frac{1}{PO^4},$$

$$\text{and the velocity at } P = u . \frac{PG}{PM} = u . \frac{AB}{PG} \propto \frac{1}{PO}.$$

(2) *A particle moves in a catenary under the action of forces acting in vertical lines; find the accelerating effect of the force and the velocity at any point.*

Let AM be the directrix, AB the ordinate at the lowest point.

$$\text{Then } PG : PM :: PM : AB \text{ and } PO = 2PG;$$

$$\therefore F = \frac{2u^2 . PG^3}{PO . PM^3} = \frac{u^2 . PM}{AB^2} \propto PM \propto PO^{\frac{1}{2}},$$

$$\text{and the velocity at } P = u . \frac{PG}{PM} = u . \frac{PM}{AB} \propto PM.$$

XXVI.

1. A body is moving in a semicircle under the action of a force tending to a point, so distant that the lines drawn from the body to that point may be considered parallel; if the centre of force be transferred to the centre of the circle, when the direction of the body's motion is perpendicular to that of the force, its magnitude at that point being unaltered, prove that the body will continue to move in the circle.

2. If a cycloid be described under the action of forces in the direction of the base, the force at any point will vary inversely as $AM . MQ$; AM, MQ being the abscissa and ordinate of the corresponding point of the generating circle.

3. A catenary is described under the action of a horizontal force, prove that the force varies as the distance from the directrix directly, and the cube of the arc from the lowest point inversely.

4. If the same parabola be described by particles when the force tends to the focus, and when it is parallel to the axis, the velocities will be equal at the points at which the forces are equal.

5. A parabola having its vertex at A and its axis coincident with AB the diameter of a semicircle, is described so as to cut the semicircle in P; prove that, if a body move in the semicircle under the action of a force perpendicular to AB, the time of moving from A to P will vary as the difference between AB and the latus rectum. Prove also, that if a second body move from A to P in the parabola in the same time under the action of a force perpendicular to its axis, and the velocities in the two curves at P be equal, the latus rectum of the parabola will be $\frac{1}{2}AB$.

PROP. IX.   PROBLEM IV.

If a body revolve in an equiangular spiral, required the law of centripetal force tending to the pole of the spiral.

Draw SY from S, the pole of the spiral, perpendicular to the tangent PY, and let PV be the chord of curvature at P, whose direction passes through S; then F', the measure of the accelerating effect of the force tending to the pole, is $\dfrac{2h^2}{SY^2.PV}$; but, if α be the angle of the spiral, $SY = SP \sin \alpha$ and $PV = 2SP$, Art. 128;

$$\therefore\ F = \frac{h^2}{\sin^2\alpha\ SP^3} = \frac{\mu}{SP^3} \propto \frac{1}{SP^3}.$$

197. *To find the velocity of a body describing an equiangular spiral under the action of a force tending to the pole.*

If $\dfrac{\mu}{SP^3}$ be the accelerating effect of the force tending to S,

$$V^2 = F.\tfrac{1}{2}PV = \frac{\mu}{SP^3}.SP; \ \therefore\ V = \frac{\mu^{\frac{1}{2}}}{SP}.$$

198. *To find the time of describing any arc of the equiangular spiral.*

Let AL be any arc, SA, SL bounding radii, P the time of describing the arc.   Then area $SAL = \tfrac{1}{4}(SA^2 \sim SL^2)\tan\alpha$, Art. 127 ;

$$\therefore\ P = \frac{2 \times \text{area } SAL}{h} = \frac{SA^2 \sim SL^2}{2h}\tan\alpha = \frac{SA^2 \sim SL^2}{2\mu^{\frac{1}{2}}\cos\alpha}\ .$$

199. *In any orbit, described under the action of a force tending to any point S, when the angle between the tangent PY and the radius SP is a maximum or minimum, the velocity is equal to the velocity in a circle at the same distance about the same force in the centre.*

For, the curve, near this point, may be considered an equiangular spiral ultimately, since the angle is constant for a short time; therefore the chord of curvature is $= 2SP$, and $V^2 = F.SP$

XXVII.

1. In different equiangular spirals, described under the action of forces tending to the poles which are equal at equal distances, shew that the angular velocity varies at any point as the force and the perpendicular on the tangent conjointly.

2. The angular velocity of the perpendicular on the tangent is equal to that of the radius.

3. The velocity of approach towards the focus, called the paracentric velocity, varies inversely as the distance.

4. A body is describing a circle, whose radius is a, with uniform velocity, under the action of a force, whose accelerating effect at any distance r is $\dfrac{\mu}{r^3}$. Prove that, if the direction of its motion be deflected inwards through any angle β without altering the velocity, the body will arrive at the centre of force after a time $\dfrac{a^2}{2\mu^{\frac{1}{2}} \sin\beta}$.

5. Deduce from the time in an equiangular spiral the time of passing from one point to another, when a body moves along a straight line with a velocity which varies inversely as the distance from a fixed point in that line.

6. A body describes an equiangular spiral in a resisting medium with uniform angular velocity under the action of a force tending to the pole; prove that the force to the pole varies as the distance and the resistance as the velocity.

7. Two particles of equal mass m, and at a distance $2a$ apart, are projected simultaneously with velocity V in the same direction perpendicular to the line joining them, the only force acting is a mutual force of attraction varying inversely as the cube of the distance between the particles, and equal at the distance $2a$ to mf. Prove that, if after a time $\sqrt{\left(\dfrac{a}{f} \cdot \dfrac{V^2 - 2af}{V^2 - af}\right)}$ one of the particles be stopped and kept at rest, the other will proceed to describe an equiangular spiral about it as pole.

8. Three particles A, B, C start from rest and move with uniform velocities, A always directing its course towards B, B towards C, and C towards A. Prove that if their velocities be proportional to b^2c, c^2a, a^2b, where a, b, c are the initial distances of B from C, C from A, and A from B respectively, they will describe similar equiangular spirals with a common pole.

PROP. X. PROBLEM V.

If a body be revolving in an ellipse, to find the law of centripetal force tending to the centre of the ellipse.

Let CA, CB be the semiaxes of the ellipse, P the position of the body at any time, PCG, DCD conjugate diameters, Q a point near P, QT, PF perpendiculars from Q and P on PC, DD'; draw QU an ordinate to PCG, QR a subtense parallel to CP.
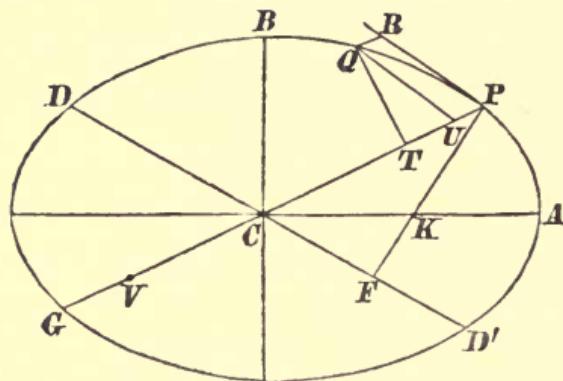
Then $F = \dfrac{2h^2}{CP^2} \cdot \dfrac{QR}{QT^2}$ ultimately.

But, by similar triangles QTU, PFC,

$$\frac{QT^2}{QU^2} = \frac{PF^2}{CP^2}, \text{ and } \frac{QU^2}{PU.UG} = \frac{CD^2}{CP^2};$$

$$\therefore \frac{QT^2}{PU.UG} = \frac{PF^2.CD^2}{CP^4} = \frac{AC^2.BC^2}{CP^4},$$

$UG = 2CP$ ultimately, and $PU = QR$;

$$\therefore \frac{QT^2}{2QR} = \frac{AC^2.BC^2}{CP^3} \text{ ultimately};$$

$$\therefore F = \text{limit of } \frac{2h^2.QR}{CP^2.QT^2} = \frac{h^2.CP}{AC^2.BC^2} \propto CP;$$

therefore the force is proportional to the distance from the centre.

Aliter.

Let CY be perpendicular on the tangent at P, and PV be the chord of curvature at P which passes through the centre $= \dfrac{2CD^2}{CP}$, Art. 79.

Then $F = \dfrac{2h^2}{CY^2 . PV} = \dfrac{h^2 . CP}{PF^2 . CD^2} = \dfrac{h^2}{AC^2 . BC^2} . CP = \mu . CP.$

Cor. 1. And conversely, if the force be as the distance, a body will revolve in an ellipse having its centre in the centre of force, or in a circle, which is a particular kind of ellipse.

Cor. 2. And the periodic times will be the same in all ellipses described by bodies about the same centre of force.

For the periodic time in any ellipse

$$= \frac{2 \times \text{area of ellipse}}{h} = \frac{2\pi AC . BC}{h},$$

and the forces, at different distances in the same or different ellipses, vary as the distance; therefore $\dfrac{h^2}{AC^2 . BC^2} = \mu$ is the same in different ellipses, therefore the periodic times in different ellipses is the same, and $= \dfrac{2\pi}{\sqrt{\mu}}.$

SCHOLIUM.

If the centre of an ellipse be supposed at an infinite distance, the ellipse will become a parabola, and the body will move in this parabola; and the force, now tending to a centre at an infinite distance, will be constant and act in parallel lines. This theorem is due to Galileo. And, if the parabola be changed into an hyperbola, by the change of inclination of the plane cutting the cone, the body will move in this hyperbola under the action of a repulsive force tending from the centre.

200. *To find the velocity in the elliptic orbit under the action of a force tending to the centre, the measure of whose accelerating effect is $\mu \times distance$.*

The velocity at $P = \dfrac{h}{CY} = \dfrac{h \cdot CD}{PF \cdot CD} = \dfrac{h \cdot CD}{AC \cdot BC} = \sqrt{\mu} \cdot CD.$

Aliter.

$(\text{Vel.})^2$ at $P = F \cdot \dfrac{PV}{2} = \mu \cdot CP \cdot \dfrac{CD^2}{CP}$; $\therefore$ vel. at $P = \sqrt{\mu} \cdot CD.$

201. *If a hyperbolic orbit be described under the action of a repulsive force tending from the centre, the force will vary as the distance, and the velocity at any point as the diameter of the conjugate hyperbola parallel to the tangent at the point.*

This may be proved exactly as in the case of the ellipse, employing the proper figure.

202. *To find the time in any arc of an elliptic orbit about a force tending to the centre.*

If P be any point of the orbit, Q the corresponding point in the auxiliary circle, time in $AP \propto$ area $ACP \propto$ area $ACQ \propto \angle ACQ$; therefore time in AP : periodic time :: $\phi : 2\pi$, if ϕ be the circular measure of $\angle ACQ$, and periodic time $= \dfrac{2\pi}{\sqrt{\mu}}$; therefore time in $AP = \dfrac{\phi}{\sqrt{\mu}}.$

203. *If, at a given point, the velocity of a body be known, and the direction of its motion; to determine the curve which the body will describe under the action of a given centripetal force, which varies as the distance from the point to which it tends.*

Let Pt be the direction of motion at P, V the velocity at P, $\mu \cdot CP$ the measure of the accelerating effect of the force tending to C. On PC produced, if necessary, take PV equal to four times the space through which a body must move from rest, under the action of the force at P continued constant, in order to acquire the given velocity V; so that $V^2 = 2\mu CP \cdot \frac{1}{4} PV.$

Draw CD parallel to Pt, a mean proportional to CP and $\frac{1}{2}PV$, and let an ellipse be constructed with CP, CD as semi-conjugate diameters, then PV is the chord of curvature at P through C.
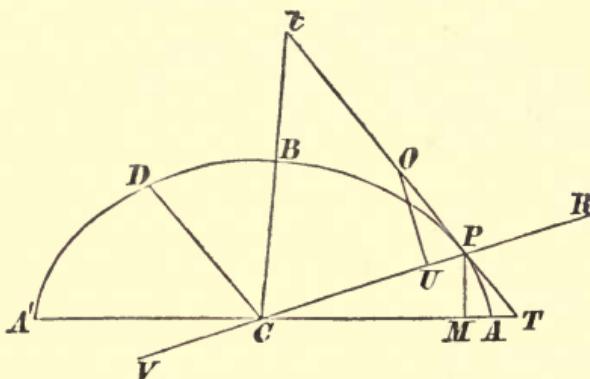
In this ellipse let a body revolve under the action of a force tending to C, whose magnitude at P is that of the given force, see Arts. 160, 162, then, when it arrives at the point P, it will be moving in the direction Pt, also the square of the velocity at $P = \mu . CD^2 = \mu . CP . \frac{1}{2}PV = V^2$, or the velocity at P, in the constructed ellipse, is V. Hence the body revolving in this ellipse is under the same circumstances as the proposed body, in all respects which can influence the motion of a body; therefore the proposed body will describe the ellipse constructed as above.

A direct solution of the problem, which is solved synthetically in this Article, is given in pages 78 and 79.

204. *Geometrical construction for the position and magnitude of the axes of the elliptic orbit, described by a body about the centre, when the velocity at a given point is known, and also the direction of motion.*

Produce CP to R, making PR a third proportional to CP and CD; bisect CR in U, and draw UC perpendicular to CR, meeting the tangent at P in O, and with centre O describe a circle passing through C, R, and cutting the tangent in T and t;

$$\therefore \; PT.Pt = CP.PR = CD^2;$$

Let TC intersect the ellipse in A, A', and draw PM parallel to the diameter conjugate to ACA';

$$\text{then } PT^2 : CD^2 :: TA . TA' : CA^2$$
$$:: CT^2 - CA^2 : CA^2 ;$$
$$\therefore PT^2 : PT . Pt :: CT^2 - CT . CM : CT . CM ;$$
$$\therefore PT : Pt :: MT : CM ;$$

hence CT is parallel to PM, and CT, Ct are in the directions of conjugate diameters; but TCt is a right angle, therefore CT, Ct being in the direction of perpendicular conjugate diameters, are the directions of the axes of the ellipse, and if PM, Pm be perpendiculars from P upon these directions, the semiaxes are mean proportionals between CM, CT, and Cm, Ct.   Q.E.F.

205. *Equations for determining the position and dimensions of the orbit.*

Let $\mu . R$ be the measure of the accelerating effect of the force at the distance $CP = R$, V the velocity, α the angle between CP and the direction of motion at the given point P. Let a, b be the semiaxes of the ellipse, ϖ the angle which CP makes with the major axis.

Then $V^2 = \mu . CD^2$ and $CD^2 + CP^2 = a^2 + b^2$;

$$\therefore a^2 + b^2 = \frac{V^2}{\mu} + R^2 \ldots\ldots\ldots\ldots\ldots(1).$$

Also $V . R \sin\alpha = h = \sqrt{\mu} . ab$;

$$\therefore ab = \frac{V . R \sin\alpha}{\sqrt{\mu}} \ldots\ldots\ldots\ldots\ldots(2),$$

and, by the properties of the ellipse,

$$\frac{R^2}{a^2} \cos^2\varpi + \frac{R^2}{b^2} \sin^2\varpi = 1 \ldots\ldots\ldots\ldots\ldots(3).$$

The equations (1), (2), and (3) determine a, b, and ϖ, whence the magnitude and position of the ellipse is determined.

We can obtain an equation for ϖ, immediately in terms of the data, as follows:

$$\left(\frac{R^2}{b^2} - 1\right) \sin^2\varpi = \left(1 - \frac{R^2}{a^2}\right) \cos^2\varpi, \text{ by (3)},$$

$$\frac{R^2}{a^2} + \frac{R^2}{b^2} = \operatorname{cosec}^2\alpha \left(1 + \frac{\mu R^2}{V^2}\right), \text{ by (1) and (2),}$$

$$\frac{R^4}{a^2 b^2} = \operatorname{cosec}^2\alpha \cdot \frac{\mu R^2}{V^2}, \text{ by (2),}$$

$$\therefore \left(\frac{R^2}{b^2} - 1\right)\left(1 - \frac{R^2}{a^2}\right) = \cot^2\alpha;$$

$$\therefore \frac{\cos^2\varpi}{\dfrac{R^2}{b^2} - 1} = \frac{\sin^2\varpi}{1 - \dfrac{R^2}{a^2}} = \frac{\sin\varpi\ \cos\varpi}{\cot\alpha}$$

$$= \frac{\cos^2\varpi - \sin^2\varpi}{\operatorname{cosec}^2\alpha \left(1 + \dfrac{\mu R^2}{V^2}\right) - 2};$$

$$\therefore \cot 2\varpi = \frac{1}{2}\tan\alpha \left(\cot^2\alpha - 1 + \operatorname{cosec}^2\alpha \cdot \frac{\mu R^2}{V^2}\right)$$

$$= \cot 2\alpha + \operatorname{cosec} 2\alpha \cdot \frac{\mu R^2}{V^2} \ \dots\dots\dots\dots(4);$$

whence ϖ is known immediately from the initial circumstances of the motion.

206. If the force be repulsive, the equations for determining a, b, ϖ will be

$$a^2 - b^2 = R^2 - \frac{V^2}{\mu} \ \dots\dots\dots\dots\dots(1),$$

$$ab = \frac{VR\sin\alpha}{\sqrt{\mu}} \ \dots\dots\dots\dots\dots(2),$$

$$\text{and } \frac{R^2}{a^2}\cos^2\varpi - \frac{R^2}{b^2}\sin^2\varpi = 1 \dots\dots\dots\dots(3).$$

The direction and magnitude of the axes of the hyperbola may be determined geometrically, by observing that the asymptotes are the diagonals of the parallelograms of which the conjugate semi-diameters are sides, and that the axes bisect the angles between the asymptotes.

207. *When a particle is acted on by any number of forces, which tend to different centres, and vary as the distances from those centres, to find the resultant attraction.*

Let $\mu . R$, $\mu' . R$ be the magnitudes of two of the forces at the distance R, A, B the centres to which they tend, P the position of a particle acted on by the forces.
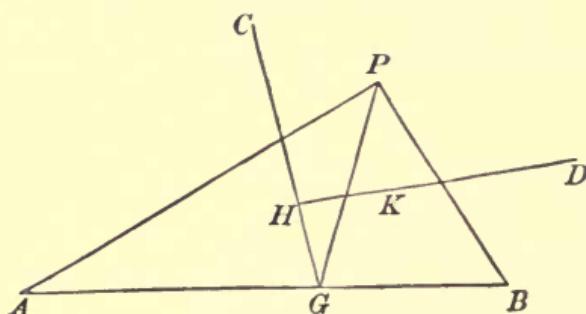
Let G be the centre of gravity of two particles at A and B whose masses are in the ratio of μ to μ', join PA, PB, PG.

The components of the force $\mu . PA$, in the directions PG, GA, are $\mu . PG$ and $\mu . GA$, and those of the force $\mu' . PB$, in the directions PG, GB, are $\mu' . PG$, and $\mu' . GB$, but $\mu . GA = \mu' . GB$, therefore the resultant of the forces tending to A and B is $(\mu + \mu') PG$, which is a single force of magnitude $(\mu + \mu') R$, at the distance R, tending to the centre of gravity of masses μ, μ' placed at A and B.

Let $\mu'' R$ be the magnitude of a force at the distance R, tending to C, the resultant attraction is that of a force tending to the centre of gravity H of particles at C and G, whose masses are in the ratio $\mu'' : \mu + \mu'$, which varies as the distance from H, and whose magnitude at the distance R is $(\mu + \mu' + \mu'') R$.

And generally, the resultant of any number of forces is a single force, tending to the centre of gravity of a system of particles, placed at the different centres, whose masses are proportional to the magnitudes of the forces at the unit distance, and whose magnitude at any distance is the sum of those of the forces at the same distance.

208. COR. 1. If every particle of a solid of any form attract with a force which varies as the mass of the particle and the distance conjointly, the resultant attraction of the solid upon any body will be the same as that of the whole mass of the solid

collected into its centre of gravity and attracting according to the same law.

209. COR. 2. If any of the forces be repulsive, as that whose centre is B, G will lie in AB or BA produced, according as μ' is greater or less than μ, and the resultant of the forces, tending to A and from B, will be $(\mu' - \mu)\ PG$ from G, or $(\mu - \mu')\ PG$ towards G.

Illustrations.

(1) *A body revolves in a circular orbit about a force which varies as the distance, and tends to the centre of the circle, and the centre of force is suddenly transferred to a point in the radius which at the moment of change passes through the body; to find the subsequent motion of the body.*

Since the force varies as the distance, and is attractive, the orbit will be an ellipse. And, since the force is a finite force, the body will move in the same direction as before, at the moment of the change. Also, the velocity will, for the same reason, be unaltered at that moment.

Let CA be the radius passing through the body at the moment of change, CB perpendicular to CA, $\mu.CA$ the force at distance CA, V the velocity in the circle.
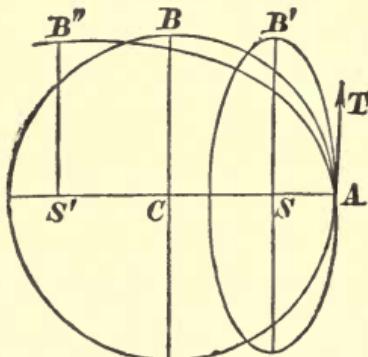
Then $V^2 = \mu.CA.CA = \mu.CA^2$; and if S, the new point to which the force tends, be in CA, let AB' be the ellipse described, SA will be one of the semi-axes of the ellipse, since A is an

apse, and, SB' being the other, if a body revolved in this ellipse round S, μ. SB'^2 would be the square of the velocity at A, that is, $\mu . SB'^2 = \mu . CA^2$, and therefore $SB' = CA = CB$; hence the magnitude and position of the two semi-axes SA and SB' are known, and the ellipse is completely determined.

The ellipse lies without the circle at A, because, the velocity being unaltered, the force has been diminished in the ratio of $SA : CA$, and therefore the curvature diminished in that ratio.

If S had been in AC produced, as at S', the force would have been increased, and the orbit AB'' would be within the circle near A.

The greatest distance from CA which the body reaches is in all cases the same for this law of force, because the component of the force perpendicular to CA is the same at the same distance from CA in whatever curve the body moves; therefore, in each orbit, the velocity being the same at A, the velocity perpendicular to AC is destroyed by the force at the same distance from AC.

(2) *A body is describing a circle about a force which varies as the distance and tends to the centre ; if the centre to which the force tends be suddenly transferred to a point in the circumference, at an angular distance of* 60° *from the position of the particle at any time, to determine the orbit described.*

The orbit is an ellipse, since the force is attractive.
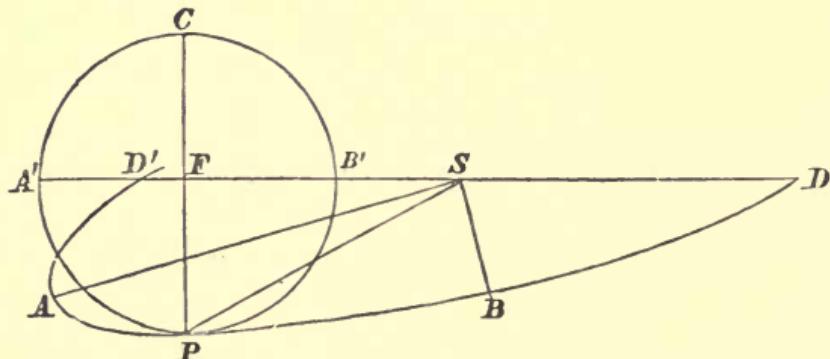
Let P be the position of the body at the instant the centre of force is transferred from C, the centre of the circle, to S, where SCP is an equilateral triangle.

The velocity at P is $\sqrt{\mu}.CP = \sqrt{\mu}.SP$; and, since it is unaltered by the change of the centre of force, the semi-diameter conjugate to SP is equal to SP.

Draw DSD' perpendicular to CP, meeting it in F, and take $SD = SD' = SP$. Construct an ellipse having SP, SD as equal conjugate semi-diameters; SA, SB the semi-axes bisect the angles PSD, $PSD.'$ The ellipse so described will be the orbit required.

Prove the following construction:

On CP as diameter describe a circle cutting SD' in B', A' SA', SB' are the lengths of the semi-axes.

Explain why the orbit is exterior to the circle.

(3) *Two bodies whose masses are m, m' revolve in an ellipse under the action of a force tending to the centre; shew that, if they be at one time at the extremities of two conjugate diameters they will always be so, and in this case find the locus of their centre of gravity.*

Let P, D be their positions at any time, CP, CD being semi-conjugate diameters. Let the ordinates MP, ND, meet the auxiliary circle in Q and R.

Since the angles ACQ, ACR are always proportional to the times, RCQ will always be a right angle; therefore the bodies will always be at the extremities of conjugate diameters.
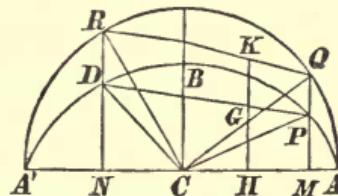
Let GH be the ordinate of their centre of gravity.
Join RQ and produce HG to meet RQ in K;

$$\therefore KH : GH = QM : PM, \text{ a constant ratio,}$$

$$\text{also,} \quad RK : KQ = DG : GP, \quad \dots\dots\dots\dots\dots;$$

hence CK is constant, or the locus of K is a circle, and the locus of G is an ellipse, whose axes are proportional to those of APD.

Shew that the semi-major axis : $CA :: (m^2 + m'^2)^{\frac{1}{2}} : m + m'$.

(4) *A body is composed of matter which attracts with a force varying as the distance ; shew that, however a particle be projected, unless it strike the body, it will describe its orbit in the same periodic time.*

This is obvious immediately from Art. 208, relating to the resultant of attracting forces.

(5) *A body moves in an ellipse under the action of a force varying as the distance ; if the velocity at any point be slightly increased in the ratio $1 + n : 1$, find the consequent changes in the axes of the ellipse.*

If, when the change takes place, the body be at the end of one of the equal conjugate diameters, shew that the eccentricity will be unaltered, and that the apse line will regrede through a small angle, whose circular measure is $\dfrac{nab}{a^2 - b^2}$.

When V is changed to $(1 + n)\ V$, CD is changed to $(1 + n)\ CD$; let the corresponding changes of a, b and ϖ be αa, βb and γ; α, β, γ, and n being so small that we may neglect their squares. Then by the equations of Art. 205,

$$(1 + \alpha)^2 a^2 + (1 + \beta)^2 b^2 = (1 + n)^2 CD^2 + R^2 = a^2 + b^2 + 2n CD^2 ;$$
$$\therefore\ \alpha a^2 + \beta b^2 = n \cdot CD^2.$$

Again $(1 + \alpha) a \cdot (1 + \beta) b = (1 + n)\ CD . R \sin\alpha = (1 + n) ab ;$
$$\therefore\ \alpha + \beta = n,\ \text{and}\ \alpha(a^2 - CD^2) = \beta\ (CD^2 - b^2),$$

$$\therefore\ \frac{\alpha}{a^2 - R^2} = \frac{\beta}{R^2 - b^2} = \frac{n}{a^2 - b^2}.$$

In the particular case $2R^2 = a^2 + b^2$, $\therefore\ \alpha = \beta = \tfrac{1}{2}n$, hence, a and b being altered in the same proportion, the eccentricity will be unaltered.

Also, $\dfrac{R^2}{a^2} \cos^2 (\varpi + \gamma) + \dfrac{R^2}{b^2} \sin^2 (\varpi + \gamma) = 1 + n$

and $\dfrac{R^2}{a^2} \cos^2 \varpi + \dfrac{R^2}{b^2} \sin^2 \varpi = 1 ;$

$$\therefore\ \left(\frac{R^2}{b^2} - \frac{R^2}{a^2} \right) \{ \sin^2 (\varpi + \gamma) - \sin^2 \varpi \} = n ;$$

$$\therefore\ \left(\frac{R^2}{b^2} - \frac{R^2}{a^2} \right) \sin (2\varpi + \gamma) \sin \gamma = n ;$$

and, since the axes bisect the angles between equal conjugate diameters, $ab = R^2 \sin 2\varpi$, therefore γ, being expressed in circular measure, $= \dfrac{nab}{a^2 - b^2}$.

(6) *In any position of a particle describing an ellipse, under the action of a force tending to the centre, the centre of force is suddenly transferred to the focus. Find the axes of the new orbit and shew that its major-axis bisects the angle between the focal distance and the major-axis of the given ellipse.*

Employing the equations of Art. 205, if α, β be the semi-axes of the new orbit, P the position of particle when the centre s transferred to S, since the semi-diameter conjugate to SP in the new orbit will be equal to CD,

$$\alpha^2 + \beta^2 = CD^2 + SP^2 = SP \cdot HP + SP^2 = 2a \cdot SP,$$
$$\text{and } SY^2 : BC^2 :: SP : HP :: SP^2 : CD^2;$$
$$\therefore \ \alpha\beta = CD \cdot SY = b \cdot SP;$$
$$\therefore \ (\alpha^2 - \beta^2)^2 = 4 \ (a^2 - b^2) \ SP^2, \text{ and } \alpha^2 - \beta^2 = 2ae SP,$$
$$\therefore \ \alpha^2 = a \ (1 + e) \ SP, \text{ and } \beta^2 = a \ (1 - e) \ SP.$$
$$\text{Also } \frac{SP^2}{\alpha^2} \cos^2 \varpi + \frac{SP^2}{\beta^2} \sin^2 \varpi = 1,$$
$$\therefore \ \frac{a \ (1 - e^2)}{SP} = (1 - e) \cos^2 \varpi + (1 + e) \sin^2 \varpi = 1 - e \cos 2\varpi;$$

therefore $2\varpi = \angle PSA$, or the major-axis of the new orbit bisects the angle between PS and the major-axis of the original orbit.

NOTE. By the construction of Art. 204, since PR is a third proportional to SP and CD, and therefore is equal to HP, the circle which determines T and t passes through H, and the arcs HT, TR are equal, that is, ST bisects the angle PSA.

XXVII.

1. Shew that the velocity in an ellipse about the centre is the same at the points whose conjugate diameters are equal as that in a circle at the same distance.

2. A body is revolving in a circle under the action of a force tending to the centre, the law of force at different distances being that the force varies as the distance; find the orbits described when the circumstances are changed at any point as follows:

i. The force is increased in the ratio of $1 : n$.

ii. The velocity is increased in the ratio $1 : n$.

iii. The force becomes repulsive, remaining of the same magnitude.

iv. The direction is changed by an impulse in the direction of the centre, measured by the velocity equal to that in the circle.

3. If a body be projected from an apse, with a velocity double of that in a circle at the same distance, find the position and magnitude of the axes of its orbit.

4. A particle is revolving in a circle acted on by a force which varies as the distance; the centre of force is suddenly transferred to the opposite extremity of the diameter through the particle and becomes repulsive; shew that the eccentricity of the hyperbolic orbit $= \frac{1}{2}\sqrt{5}$.

5. A body is moving under the action of a force tending to a fixed centre, and varying as the distance. The force suddenly ceases, and after an interval commences to act again. Prove that the radii of curvature of the orbit at the points where the body ceases and recommences to be attracted are equal.

6. A body moves in an ellipse about a centre of force in the centre, and its velocity is observed when it arrives at its greatest distance, and again after a lapse of one-third of its periodic time. If these velocities be in the ratio of $2 : 3$, prove that the eccentricity of the ellipse will be $\sqrt{\frac{3}{5}}$.

7. The particles of which a rectangular parallelepiped is composed attract with a force which varies as the distance, and a body is projected so as to describe a curve on one of the faces supposed smooth; find the periodic time.

8. An elastic ball, moving in an ellipse about the centre, on arriving at the extremity of the minor axis strikes directly another ball at rest; find the orbits described by both bodies.

9. A body is projected in a direction making an angle $\cos^{-1} \dfrac{1}{\sqrt{3}}$ with the distance from a point to which a force tends, varying as the distance from it, and the velocity $= \sqrt{\frac{2}{3}} \times$ velocity in the circle at the same distance; prove that one axis is double of the other and that the inclination of the major axis to the distance is $\frac{1}{2} \cos^{-1} \frac{1}{3}$.

10. From points in a line CA between C and A particles are projected at right angles to CA with velocities proportional to their distances from A, C being a centre to which the force tends, and the force varying as the distance; find the ellipse of greatest area which is described.

11. Two particles are projected in parallel directions from two points in a straight line passing through a centre of force, the acceleration towards which varies as the distance, with velocities proportional to their distances from that centre. Prove that all tangents to the path of the inner cut off, from that of the outer, arcs described in equal times.

12. An hyperbola and its conjugate are described by particles round a force in the centre. They are at an apse at the same instant; shew that they will always be at the extremities of conjugate diameters. Also if v, v' be their velocities, $v^2 - v'^2 = \mu (a^2 - b^2)$.

13. An ellipse and an hyperbola have the same centre and foci. They are described by particles, under the action of forces in the centre of equal intensity. If a, a' be their semi-transverse axes, the square of the velocity of each body at a point where the curves cut will be $\mu \ a^2 - a'^2$).

14. If any number of particles be moving in an ellipse about a force in the centre, and the force suddenly cease to act, shew that, after the lapse of $\dfrac{1}{2\pi}$ of the period of a complete revolution, all the particles will be in a similar, concentric, and similarly situated ellipse.

15. A particle is describing an ellipse under the action of a force tending to the centre. Prove that its angular velocity about a focus is inversely proportional to its distance from that focus.

XXVIII.

1. CX, CY are straight lines inclined at any angle, and a force tends to C, and varies as the distance from C. If from various points in CY different particles are projected parallel to CX at the same moment, and with the same velocity, they will all arrive at CX at the same time and place; and they will also do so, if the force cease to act for any interval of time.

2. A number of particles move in hyperbolas, under the action of the same repulsive force from their common centre. Shew that, if the transverse axes coincide, and the particles start from the vertex at the same instant, they will always lie in a straight line